SOLUTION KEY

EE301 Signals and Systems **In-Class Exam**
Exam 2 **Thursday, Mar. 29, 2012**

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 3,4 with emphasis on Chap. 4
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

True False Questions. Circle the True (T) or False (F) for each part below.

(T) (F) Let $X(\omega)$ be the Fourier Transform of a signal $x(t)$; as you vary ω, a plot of $|X(\omega)|^2$ reveals how the energy of the signal is distributed as a function of frequency.

(T) (F) Two distinctly different signals can have the same Fourier Transform.

(T) (F) $x(t)$ and $x(t - t_o)$ have the same energy distribution as a function of time.

(T) (F) $x(t - t_o)$ can be obtained by passing $x(t)$ thru an LTI system with impulse response $h(t) = \delta(t - t_o)$.

(T) (F) $x(t)$ and $e^{j\omega_o t}x(t)$ always have the same energy distribution as a function of frequency.

(T) (F) Forming the product of a baseband signal with a high-frequency sinewave places the signal in another frequency band

(T) (F) Multiplying by the independent variable in one domain (time or frequency) causes differentiation with respect to the independent variable in the other domain.

(T) (F) Multiplying by the independent variable in one domain (time or frequency) causes multiplication by the independent variable in the other domain.

(T) (F) One of the most important practical implications of the convolution (in time) property of the Fourier Transform (convolution in time leads to multiplication in the frequency domain) is frequency selective linear filtering.

(T) (F) For any input signal, the energy distribution (in either the time or frequency domain) is the same at both the input and output of an LTI system.

Problem 2. Short Workout Problems Using Fourier Transform Properties.
Problem 2 (a). You are given that the impulse response of an ideal Hilbert Transformer has the frequency response (Fourier Transform) given below.

$$h(t) = \frac{1}{\pi t} \longleftrightarrow H(\omega) = \begin{cases} j, & \text{for } \omega < 0 \\ -j, & \text{for } \omega > 0 \end{cases}$$

Just view the above as a Fourier Transform Pair, and use one or more of the Fourier Transform properties to determine the Fourier Transform of

$$x(t) = sgn(t) = \begin{cases} -1, & \text{for } t < 0 \\ 1, & \text{for } t > 0 \end{cases}$$

Write your expression for $X(\omega)$ in the space directly below:

Duality Property: $x_0(t) = \frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X_0(\omega) = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases}$

$X_0(t) = \begin{cases} j, & t < 0 \\ -j, & t > 0 \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x_0(-\omega)$

$= 2\pi \frac{1}{\pi(-\omega)}$

Multiply both sides by j:

$x(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} -\frac{2j}{\omega} = X(\omega)$

Problem 2 (b). Given the Fourier Transform pair below, write your expression for $Y(\omega)$ in the space directly below.

Time-Scale Property:

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$a = \frac{1}{T}$

$$x(t) = \cos(\pi t)\,\text{rect}(t) \longleftrightarrow X(\omega) = \frac{2\pi\cos(\frac{\omega}{2})}{\pi^2 - \omega^2}$$

Determine the Fourier Transform of the signal below in terms of T.

$$y(t) = \cos\left(\pi\frac{t}{T}\right)\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow Y(\omega) = ??$$

$$Y(\omega) = T\,\frac{2\pi\cos\left(T\frac{\omega}{2}\right)}{\pi^2 - (T\omega)^2}$$

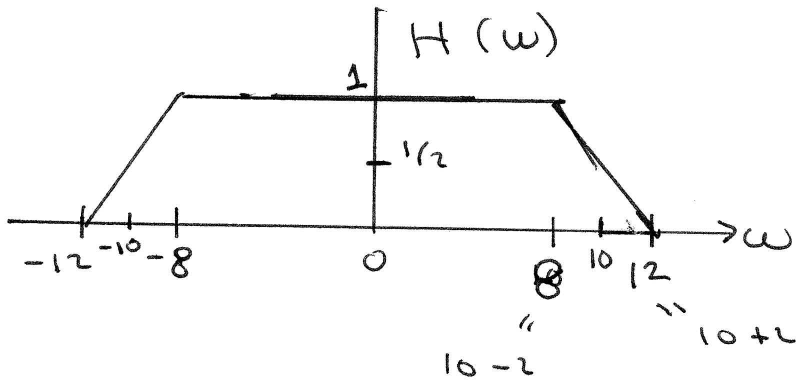
Problem 2 (c). Consider an LTI system with impulse response

$$h(t) = \frac{\pi}{2}\,\frac{\sin(2t)}{\pi t}\,\frac{\sin(10t)}{\pi t} \tag{1}$$

Determine the output $y(t)$ for the input $x(t)$ given below, which is the Fourier Series expansion for a periodic sawtooth waveform with period $T = \pi$.

$$x(t) = \sum_{k=-\infty}^{-1} \frac{j(-1)^k}{k\pi}\, e^{jk2t} + \sum_{k=1}^{\infty} \frac{j(-1)^k}{k\pi}\, e^{jk2t}$$

Show work and write your expression for $y(t)$ in the space directly below.



$H(10) = \frac{1}{2} = H(-10)$

$k = 5$

Frequencies in $x(t)$: $\omega = 2k, \quad -\infty < k < \infty$

$k=0$

$$\ldots \underbrace{-14, -12,}_{\substack{\text{rejected}\\ \text{since } H(\omega)=0\\ \text{for } \omega \le -12}} -10, \underbrace{-8, -6, -4, -2, 0, 2, 4, 6, 8}_{\substack{H(\omega)=1 \text{ for all}\\ \text{these frequencies}}}, 10, \underbrace{12, 14,}_{\substack{\text{rejected}\\ \text{since } H(\omega)=0\\ \text{for } \omega \ge 12}} \ldots$$

$$y(t) = \frac{j(-1)^{-5}}{(-5)\pi} \overbrace{\left(\frac{1}{2}\right)}^{H(-10)} e^{-j10t} + \sum_{k=-4}^{4} \frac{j(-1)^k}{k\pi} e^{j2kt} + \frac{j(-1)^5}{5\pi} \underbrace{\left(\frac{1}{2}\right)}_{H(10)} e^{j10t}$$

Workout Problem 3

(a) Let $H_0(\omega)$ be the Fourier Transform of the impulse response $h_0(t)$ defined below.

$$h_0(t) = 2\frac{\sin(5t)}{\pi t}\sin(5t) \tag{2}$$

Note that $h_0(t)$ is both real-valued and odd-symmetric as a function of time. Thus, $H_0(\omega)$ is purely imaginary-valued and odd-symmetric as a function of frequency. Plot $H_0(\omega)$ in the space provided. Note that the vertical axis values have the multiplicative scalar $j = \sqrt{-1}$ factored into them.

(b) Determine and plot, in the space provided, the Fourier Transform $X(\omega)$ of the signal $x(t)$:

$$x(t) = \frac{\sin(10t)}{\pi t} + \frac{1}{2}\left\{\frac{\sin(10(t-\frac{\pi}{10}))}{\pi(t-\frac{\pi}{10})} + \frac{\sin(10(t+\frac{\pi}{10}))}{\pi(t+\frac{\pi}{10})}\right\}$$

(c) Determine and plot the Fourier Transform for the signal $y_0(t)$ defined below, where $\hat{x}_0(t) = x(t) * h_0(t)$ with $h_0(t)$ and $x(t)$ defined in parts (a) and (b), respectively. Plot $Y_0(\omega)$ in the space provided.

$$y_0(t) = x(t) + j\hat{x}_0(t) \quad \text{where:} \quad \hat{x}_0(t) = x(t) * h_0(t)$$

(d) Determine and plot the Fourier Transform for the signal $y_1(t)$ defined below, where $\hat{x}_1(t) = x(t) * h_1(t)$ with $h_1(t) = \frac{1}{\pi t}$ and $x(t)$ defined in part (b). Plot $Y_1(\omega)$ in the space provided.

$$y_1(t) = x(t) + j\hat{x}_1(t) \quad \text{where:} \quad \hat{x}_1(t) = x(t) * \frac{1}{\pi t}$$

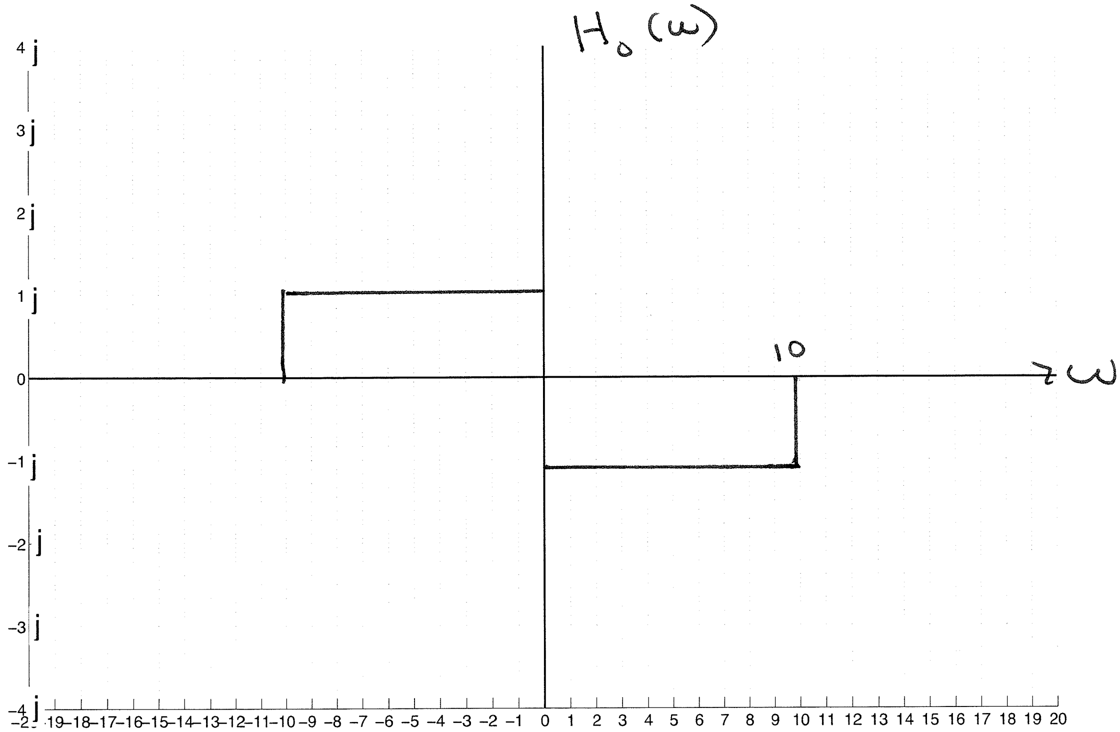
(e) Determine and plot the Fourier Transform for the signal $z_0(t)$ defined below where, as defined previously, $\hat{x}_0(t) = x(t) * h_0(t)$ with $h_0(t)$ and $x(t)$ defined in parts (a) and (b), respectively. Plot $Z_0(\omega)$ in the space provided.

$$z_0(t) = x(t)\cos(30t) - \hat{x}_0(t)\sin(30t)$$

(f) Determine and plot the Fourier Transform for the signal $z_1(t)$ defined below where, as defined previously, $\hat{x}_1(t) = x(t) * h_1(t)$ with $h_1(t) = \frac{1}{\pi t}$ and $x(t)$ defined in part (b). Plot $Z_1(\omega)$ in the space provided.

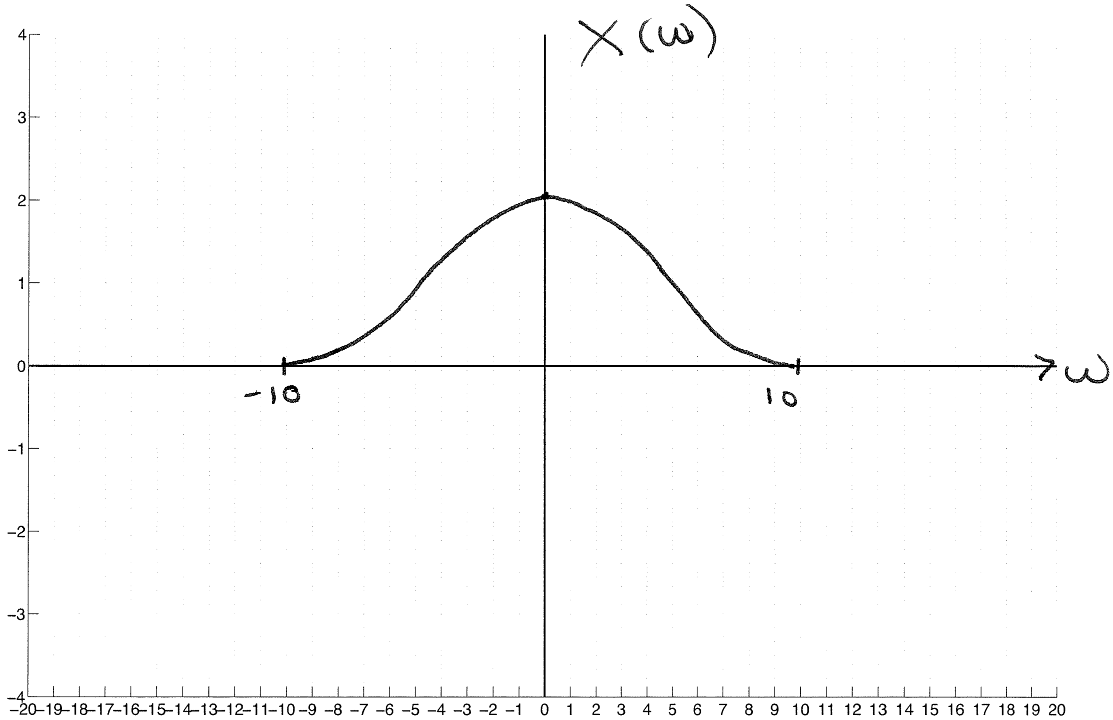
$$z_1(t) = x(t)\cos(30t) + \hat{x}_1(t)\sin(30t)$$

Plot your answer to Problem 3 (a) here. Show work below.



$$2\,\frac{\sin(5t)}{\pi t}\,\sin(5t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j}\,\mathrm{rect}\left(\frac{\omega-5}{10}\right) - \frac{1}{j}\,\mathrm{rect}\left(\frac{\omega+5}{10}\right)$$

Plot your answer to Problem 3 (b) here. Show work below.



$$\frac{\sin(10t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{20}\right) \qquad x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$$

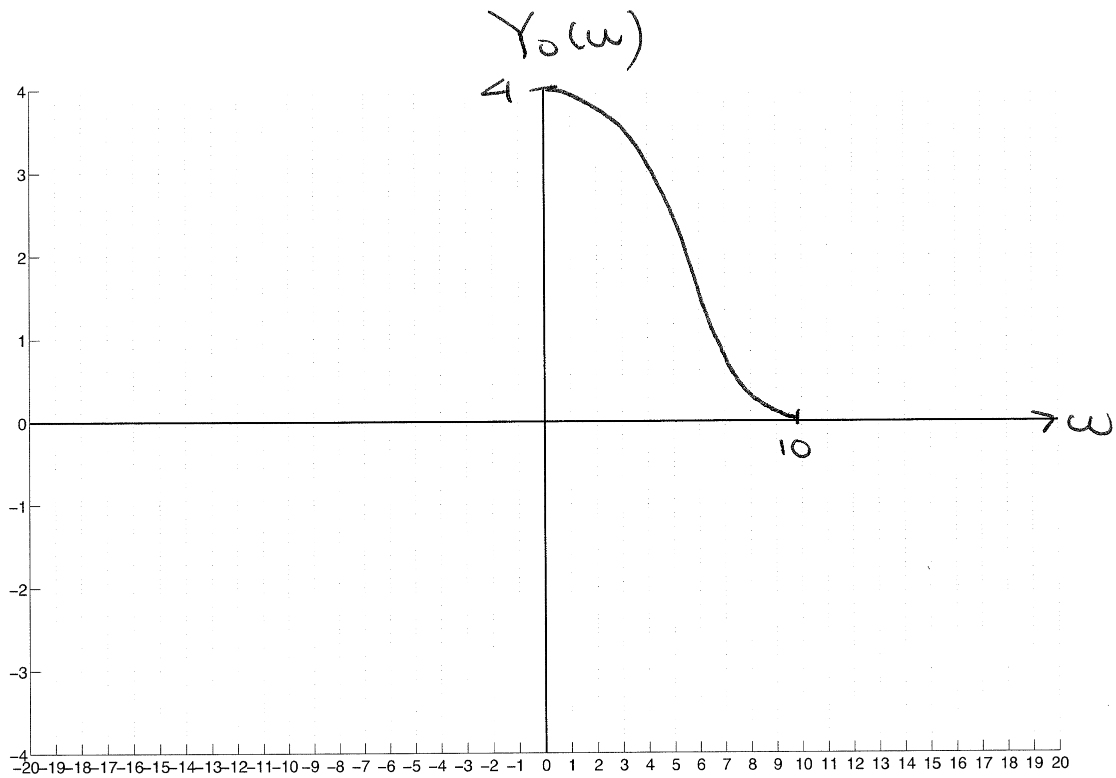
$$X(\omega) = \mathrm{rect}\left(\frac{\omega}{20}\right)\left\{1 + \frac{1}{2}e^{-j\frac{\pi}{10}\omega} + \frac{1}{2}e^{j\frac{\pi}{10}\omega}\right\}$$

$$= \underbrace{\mathrm{rect}\left(\frac{\omega}{20}\right)}_{=1 \text{ for } |\omega|<10}\left\{1 + \cos\left(\frac{\pi}{10}\omega\right)\right\}$$

at $\omega = 10$

$$\cos\left(\frac{\pi}{10}10\right) = \cos(\pi) = -1$$

Plot your answer to Problem 3 (c) here. Show work below.



Since $H_0(\omega) = -j$ for $0 < \omega < 10$

and $+j$ for $-10 < \omega < 0$

the negative frequency content in $X(\omega)$ is removed

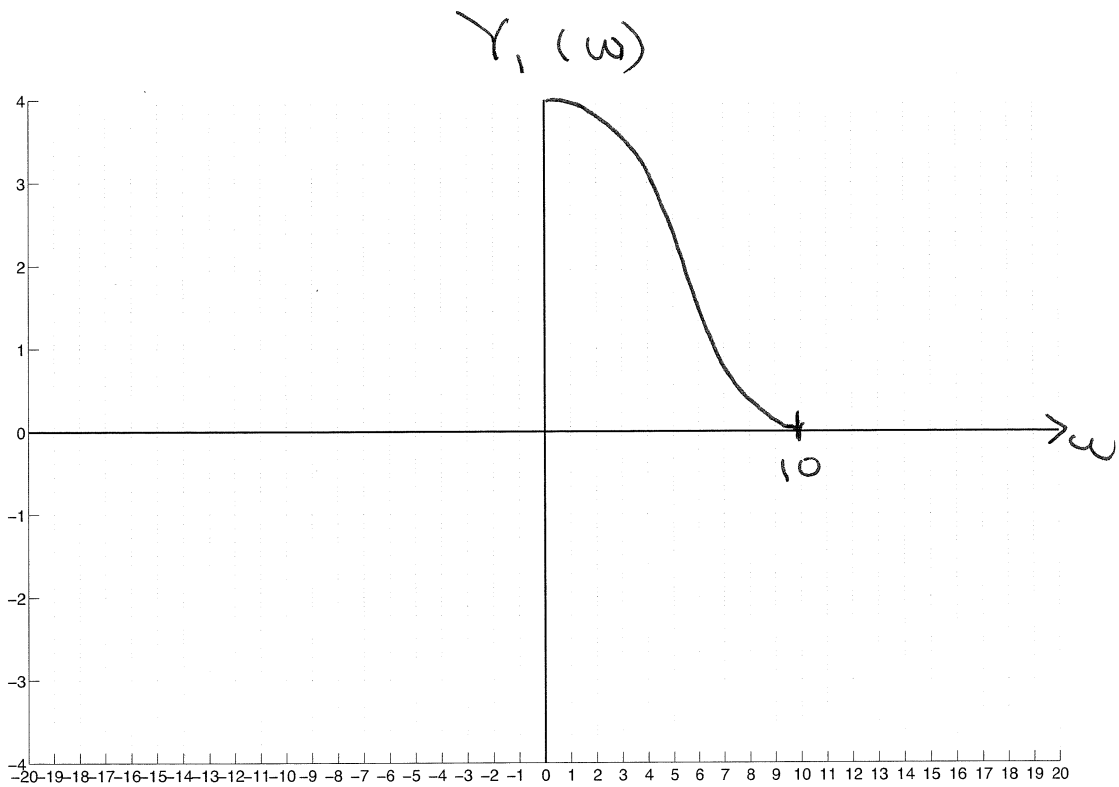
That is,

$$Y_0(\omega) = X(\omega) + j\,H_0(\omega)\,X(\omega)$$

$$= X(\omega)\{1 + j\,H_0(\omega)\}$$

$$= X(\omega) = \begin{cases} 1 + j(j) = 0, & \text{for } -10 < \omega < 0 \\ 1 + j(-j) = 2, & \text{for } 0 < \omega < 10 \end{cases}$$

Plot your answer to Problem 3 (d) here. Show work below.



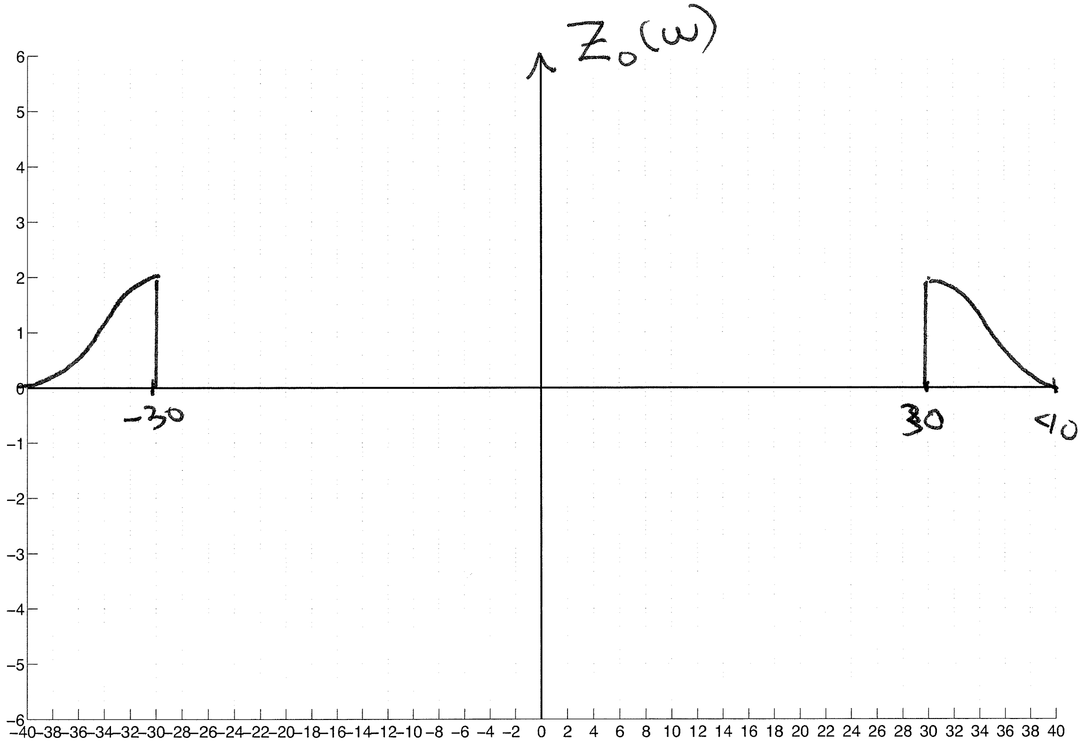
From Prob. 2 (and class notes)

$$h_1(t) = \frac{1}{\pi t} \overset{\mathcal{F}}{\Longleftrightarrow} H_1(\omega) = \begin{cases} +j, & \omega < 0 \\ -j, & \omega > 0 \end{cases}$$

Observe: $\begin{cases} x(t) * h_0(t) = x(t) * h_1(t) \\ \text{since } X(\omega) = 0 \text{ for } |\omega| > 10 \end{cases}$

Thus, same answer as 3(c)

Plot your answer to Problem 3 (e) here. Show work below.

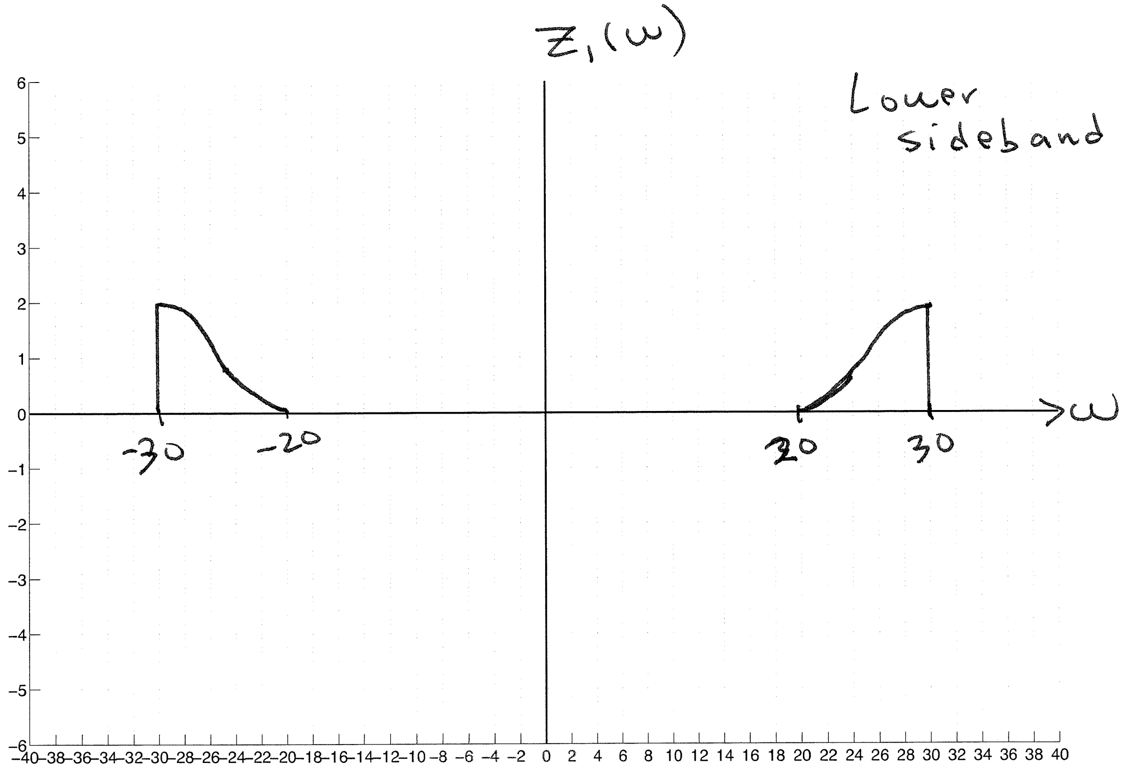


Based on class notes,

$$z_o(t) = \mathrm{Re}\left\{ \tilde{x}(t)\, e^{j30t} \right\}$$

$$= \mathrm{Re}\left\{ \left(x(t) + j\, x(t) * \underset{\substack{\text{or} \\ h_1(t)}}{h_o(t)} \right) e^{j30t} \right\}$$

Plot your answer to Problem 3 (f) here. Show work below.



This was a curve ball to see if you could figure this out. :)

$$z_1(t) = \mathrm{Re}\left\{ \underbrace{\left(x(t) - j\, x(t) * h_1(t) \right)}_{\text{blanks out positive frequencies since}} e^{j30t} \right\}$$

blanks out <u>positive</u> frequencies since

$$X(\omega)\{1 - jH_1(\omega)\}$$

where: $1 - jH_1(\omega) = 1 - j(j) = 2, \quad \omega < 0$

$1 - j(-j) = 0, \quad \omega > 0$

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(\omega)$
		$y(t)$	$Y(\omega)$
4.3.0	Duality	$X(t)$	$2\pi\, x(-\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
4.3.3	Conjugation	$x^*(t)$	$X^*(-\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(\omega)Y(\omega)$
4.5	Multiplication	$x(t)y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}X(\omega)*Y(\omega) =$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\theta)Y(\omega - \theta)\,d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(\omega) = X^*(-\omega) \\ \mathcal{R}e\{X(\omega)\} = \mathcal{R}e\{X(-\omega)\} \\ \mathcal{I}m\{X(\omega)\} = -\mathcal{I}m\{X(-\omega)\} \\ \lvert X(\omega)\rvert = \lvert X(-\omega)\rvert \\ \sphericalangle X(\omega) = -\sphericalangle X(-\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(\omega)$ purely imaginary and od
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(\omega)\}$ $j\mathcal{I}m\{X(\omega)\}$
Initial Value Theorems:		$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$ $X(0) = \int_{-\infty}^{\infty} x(t)\,dt$	
4.3.7	Parseval's Relation for Aperiodic Signals		

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$