SOLUTION KEY

EE301 Signals and Systems — Spring 2014
Exam 2 — Thursday, Mar. 27, 2014

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 3,4 with emphasis on Chap. 4
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You can NOT do work on the back of a page unless permission is granted.
No work on the back of a page will be graded unless permission is granted.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

Yes or No Question with Explanation. Circle Yes or No for each question below, and briefly **explain** your answer in the space provided. You must cite at least one Fourier Transform Property in your answer for each part. The explanation you give is much more important than your "yes" or "no" answer.

Yes No The signal $y(t)$ is formed as $y(t) = x(t) + j\left(x(t) * \frac{1}{\pi t}\right)$. The Fourier Transform of $y(t)$ is denoted $Y(\omega)$. Is $Y(\omega)$ an even-symmetric function of frequency?

$$Y(\omega) = \begin{cases} 0, & \omega < 0 \\ 2X(\omega), & \omega > 0 \end{cases}$$

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$Y(\omega)$ is definitely not even-symmetric

Yes No Again, the signal $y(t)$ is formed as $y(t) = x(t) + j\left(x(t) * \frac{1}{\pi t}\right)$. The Fourier Transform of $y(t)$ is denoted $Y(\omega)$. Does $\int_{-\infty}^{0} |Y(\omega)|^2 = 0$?

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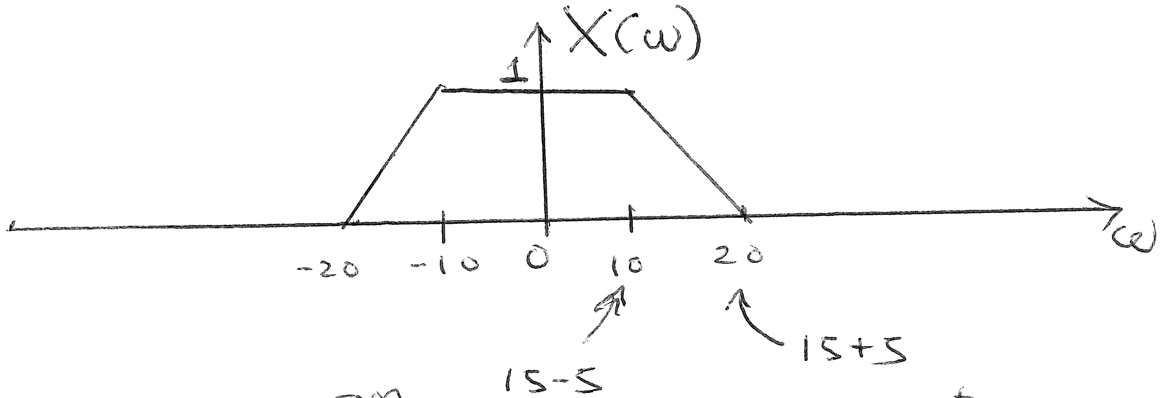
See above: $Y(\omega) = 0$ for $\omega < 0$

so $\int_{-\infty}^{0} |Y(\omega)|^2 \, d\omega = 0$

Problem 2 (a). Determine the numerical value of the area $A = \int_{-\infty}^{\infty} x(t)dt$ for the signal defined below. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$x(t) = \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t} \right\} \tag{1}$$



$$X(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\,dt = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$

answer: $X(0) = 1$

Problem 2 (b). Compute the energy $E = \int_{-\infty}^{\infty} x^2(t)dt$ of the signal below. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$x(t) = \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t} \right\} \tag{2}$$

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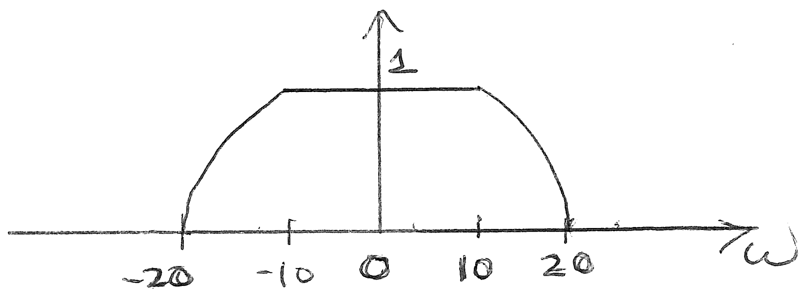
$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$



Area under $|X(\omega)|^2$ $= 2(10)(1) + 2\int_{10}^{20} \frac{1}{10^2}(\omega-20)^2 \, d\omega$

$$= 2\int_0^{20} |X(\omega)|^2 d\omega = 20 + \frac{2}{100} \left. \frac{(\omega-20)^3}{3} \right|_{10}^{20}$$

$$= 20 + \frac{2}{300} \left\{ 10^3 \right\}$$

$$= 20 + \frac{1000}{300} = 20 + \frac{20}{3}$$

$$= (20)\ \frac{4}{3} \left(\frac{1}{2\pi} \right)$$

$= 40/(3\pi)$

$\sigma_1 = 3 \qquad \sigma_2 = 4$ 10

Problem 2 (c). The Gaussian pulse $x(t) = e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^2$. Determine a simple, closed-form expression for the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$e^{-\frac{t^2}{2\cdot 3^2}} * e^{-\frac{t^2}{2\cdot 4^2}} \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\,3\, e^{-\frac{\omega^2 3^2}{2}} \; \sqrt{2\pi}\,4\, e^{-\frac{\omega^2 4^2}{2}}$$

$$= 2\pi\,(12)\,\frac{\sqrt{2\pi}\,5}{\sqrt{2\pi}\,5}\, e^{-\frac{\omega^2}{2}(5^2)}$$

$$\sqrt{2\pi}\,\frac{12}{5}\, e^{-\frac{t^2}{2\cdot 5^2}} \xleftrightarrow{\mathcal{F}}$$

Problem 2 (d). The Gaussian pulse $x(t) = e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4$. Determine the numerical value of the area $A = \int_{-\infty}^{\infty} y(t)dt$ under the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$Y(0) = \int_{-\infty}^{\infty} y(t)\,dt$$

$$Y(0) = H(0)\,X(0)$$

$$= \left(\sqrt{2\pi}\ 3\ e^{0}\right)\left(\sqrt{2\pi}\ 4\ e^{0}\right)$$

$$= 2\pi\,(12)$$

$$= 24\pi$$

Problem 2 (e). The Gaussian pulse $x(t) = e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^{2}$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^{2}$. Determine the numerical value of the energy $E_y = \int_{-\infty}^{\infty} y^2(t)dt$ under the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

From 2 (c): $y(t) = \sqrt{2\pi}\,\frac{12}{5}\, e^{-\frac{t^2}{2\cdot 5^2}}$

$$E_y = \int_{-\infty}^{\infty} y^2(t)\,dt$$

$$= 2\pi \left(\frac{12}{5}\right)^2 \underbrace{\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\frac{5^2}{2}}}\,dt}_{\sqrt{2\pi}\,\frac{5}{\sqrt{2}}}$$

$$= 2\pi\,\frac{144}{25}\cdot\sqrt{\pi}\;5$$

$$= \left(\pi^{3/2}\right)\frac{288}{5}$$

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Problem 2 (f). Determine the signal $x(t)$ that has the Fourier Transform $X(\omega) = \frac{1}{\pi\omega}$

$$x(t) = ?? \longleftrightarrow X(\omega) = \frac{1}{\pi\omega}$$

where $ln(t)$ is the natural logarithm of t such that Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$\frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} -j \,\mathrm{sgn}(\omega)$$

Duality dictates:

$$-j \,\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \frac{1}{\pi(-\omega)}$$

$$-j \,\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} -\frac{2}{\omega}$$

$$\left(\frac{1}{2\pi}\right) j \,\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{\omega} \left(\frac{1}{2\pi}\right)$$

$$\frac{j}{2\pi} \,\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\pi\omega}$$

Problem 2 (g). EXTRA CREDIT. *Save this problem for last.* +10

The signal $y(t) = x(t) + j\left(x(t) * \frac{1}{\pi t}\right)$ can be expressed in terms of convolution as $y(t) = x(t) * h(t)$. What is $h(t)$? Plot $H(\omega)$.

$$x(t) = x(t) * \delta(t)$$

$$y(t) = x(t) * \delta(t) + j\, x(t) * \frac{1}{\pi t}$$

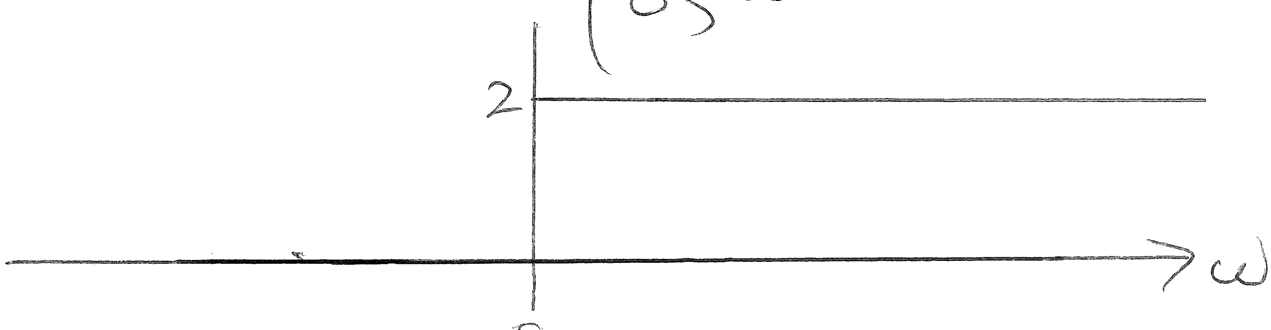
$$= x(t) * \left\{ \delta(t) + j \frac{1}{\pi t} \right\}$$

$$h(t) = \delta(t) + j \frac{1}{\pi t}$$

$$H(\omega) = 1 + j\left(-j \operatorname{sgn}(\omega)\right)$$

$$= 1 + \operatorname{sgn}(\omega)$$

$$= \begin{cases} 2, & \omega > 0 \\ 0, & \omega < 0 \end{cases}$$



Workout Problem 3.

(a) For the signal $s(t)$ below, plot the Fourier Transform $S(\omega)$ which is purely real-valued:

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$$s(t) = \frac{1}{2j}\left\{\frac{\sin(10(t-\frac{\pi}{10}))}{\pi(t-\frac{\pi}{10})} - \frac{\sin(10(t+\frac{\pi}{10}))}{\pi(t+\frac{\pi}{10})}\right\} \tag{3}$$

(b) An AM signal $r(t)$ is formed from $s(t)$ above as prescribed below, where $k = 0.1$.

$$r(t) = [1 + k\, s(t)]\cos(35t) \tag{4}$$

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The signal $r(t)$ above is applied to a square-law device, followed by amplification, to form $x(t) = 10r^2(t)$

$$\begin{aligned} x(t) &= 10\,\{[1 + k\, s(t)]\cos(35t)\}^2 \\ &= 10\,[1 + k\, s(t)]^2\cos^2(35t) \end{aligned} \tag{5}$$

Plot the Fourier Transform of $x(t)$ denoted $X(\omega)$, IGNORE as negligible any term which is scaled by $k^2 = 0.01$. Recall $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}cos(2\theta)$.

(c) Consider an LTI system with impulse response

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$$h(t) = \left\{\frac{\pi}{5}\,\frac{\sin(5t)}{\pi t}\,\frac{\sin(15t)}{\pi t}\right\} \tag{6}$$

Determine and plot the frequency response, $H(\omega)$, the Fourier Transform of $h(t)$.

(d) For the LTI system with this impulse response, determine the output $y(t)$ for the input $x(t)$ above. Plot $Y(\omega)$. Again, ignore any term which has scaled by $k^2 = 0.01$ as negligible.

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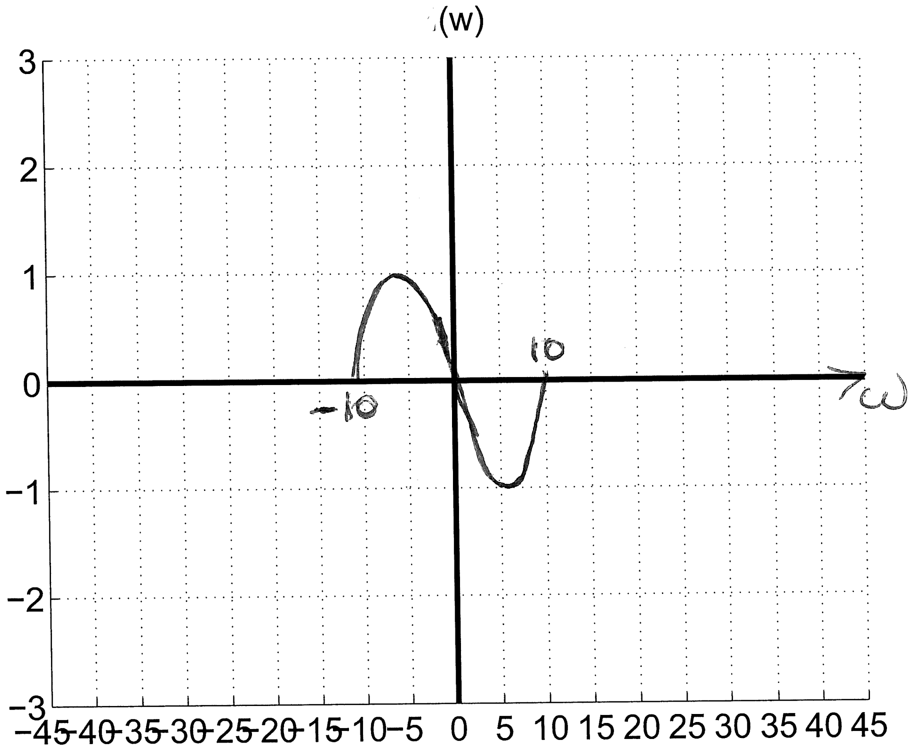
$$y(t) = x(t) * h(t)$$

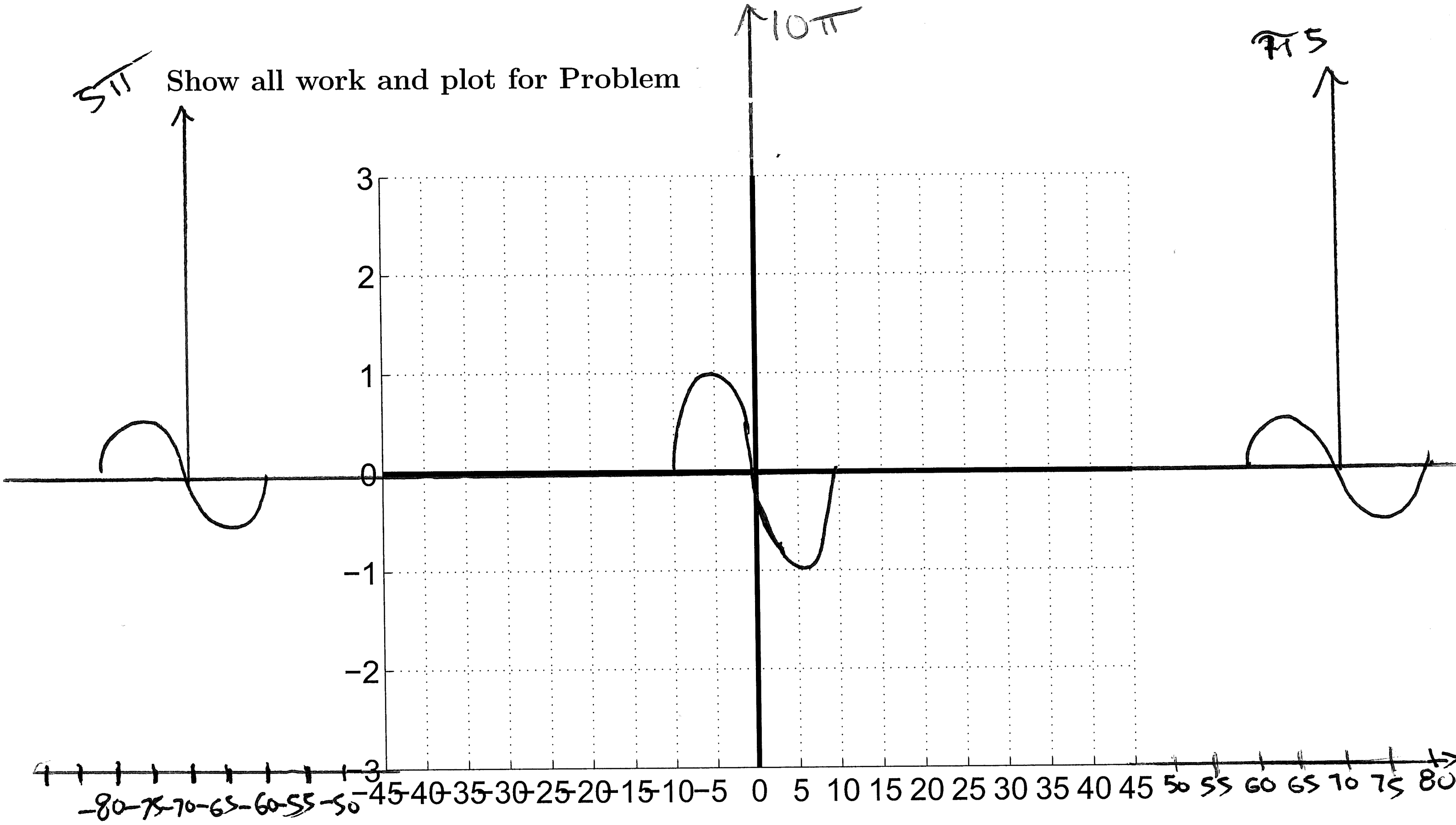
Show all work and plot for Problem 3 (a.).



$$S(\omega) = \text{rect}\left(\frac{\omega}{20}\right)\frac{1}{2j}\left\{e^{-j\omega\frac{\pi}{10}} - e^{+j\omega\frac{\pi}{10}}\right\}$$

$$= -\sin\left(\frac{\pi}{10}\omega\right)\text{rect}\left(\frac{\omega}{20}\right)$$

Show all work and plot for Problem



$$x(t) = 10\left(1 + k\,s(t)\right)^2 \cos^2(35t)$$

$$= 10\left(1 + 2k\,s(t) + k^2 s^2(t)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(70t)\right)$$

assume $k^2 = .01$ is negligible

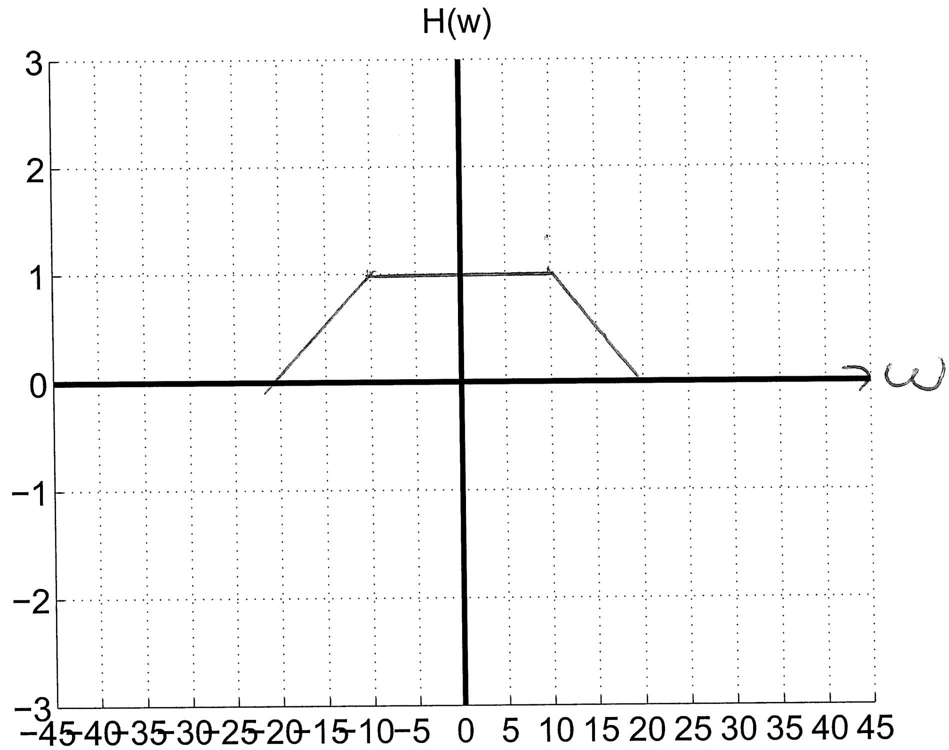
$$x(t) = 5\left(1 + 2(.1)\,s(t)\right)\left(1 + \cos(70t)\right)$$

$$= \left(5 + s(t)\right)\left(1 + \cos(70t)\right)$$

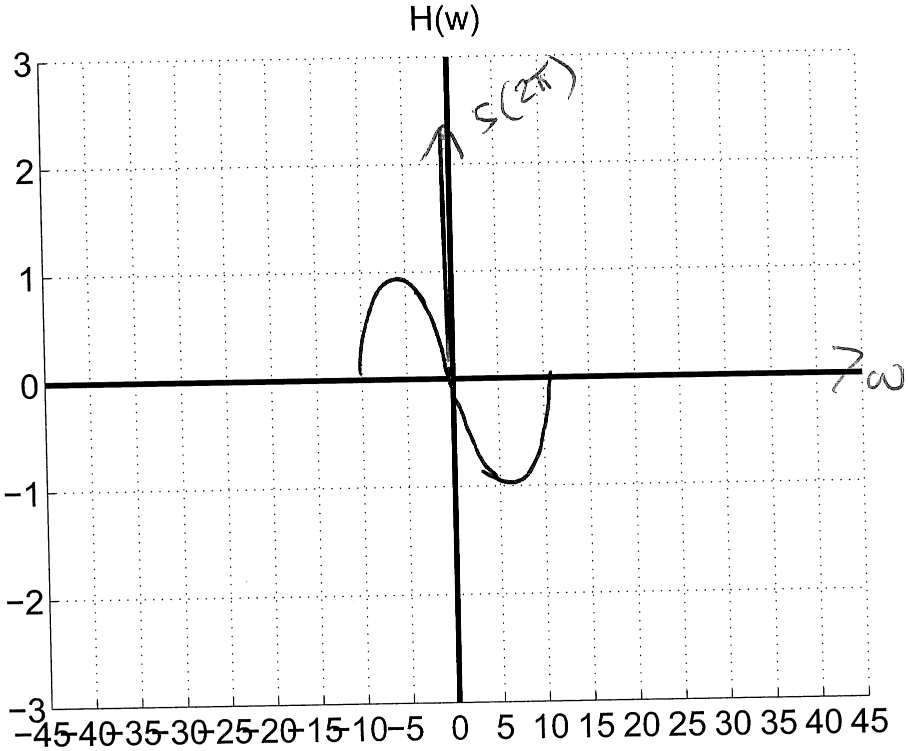
$$= 5 + s(t) + 5\cos(70t) + s(t)\cos(70t)$$

$$X(\omega) = 10\pi\,\delta(\omega) + S(\omega) + 5\pi\,\delta(\omega-70) + 5\pi\,\delta(\omega+70) + \frac{1}{2}S(\omega-70) + \frac{1}{2}S(\omega+70)$$

Show all work and plot for Problem 3 (c.).



Show all work and plot for Problem 3 (d.).



$y(t) = 5 + s(t)$