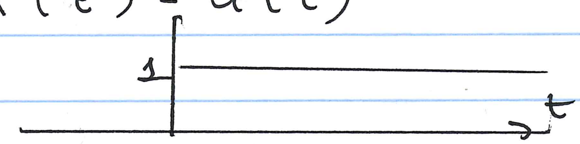
ECE 301 Exam 1 Summary

Various Systems:

$y(t) = x(at)$ L not TI not causal	$y(t) = x(t)\, g(t)$ L not TI examples: $g(t) = t$ $g(t) = \cos(\omega_0 t)$	$y(t) = x^2(t)$ not L, TI nonlinear

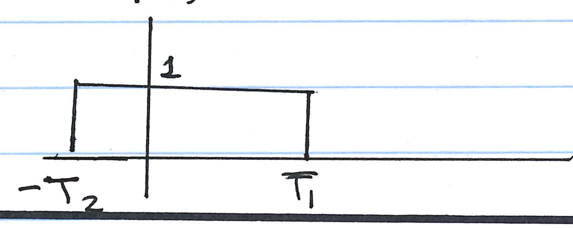
$y(t) = x(t - t_0)$ LTI impulse response: $h(t) = \delta(t - t_0)$

$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ LTI $h(t) = u(t)$

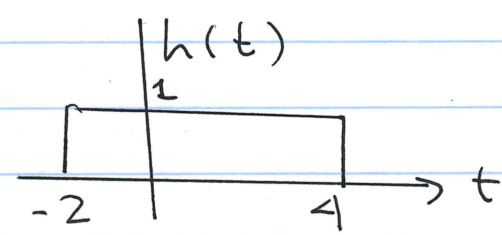


$y(t) = \int_{t-T_1}^{t+T_2} x(\tau)\, d\tau$ LTI $h(t) = u(t+T_2) - u(t-T_1)$

$T_1 > 0 \quad T_2 \geq 0$



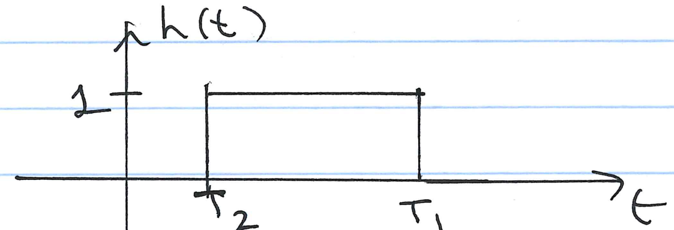
Example e.g. $y(t) = \int_{t-4}^{t+2} x(\tau)\, d\tau$



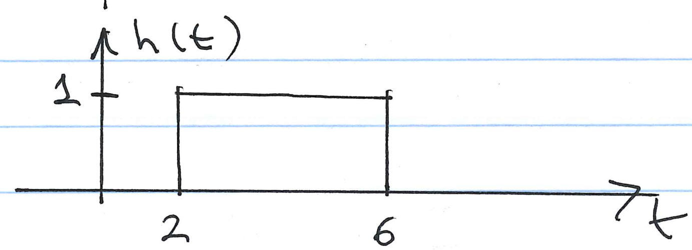
$y(t) = \int_{t-T_1}^{t-T_2} x(\tau)\, d\tau$

$T_1 > 0 \quad T_2 > 0$

$T_1 > T_2$



Example: $y(t) = \int_{t-6}^{t-2} x(\tau)\, d\tau$



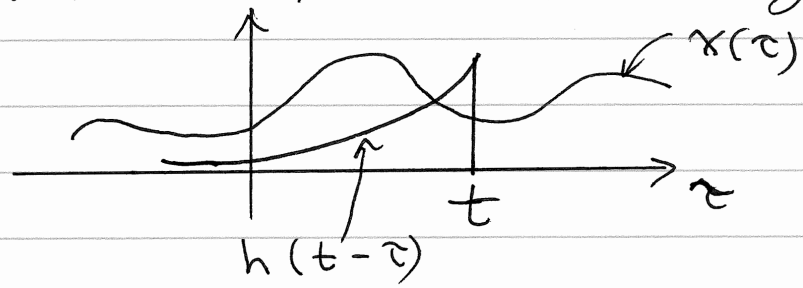
Executive Summary Add-Ons :)

System 1:

$$y(t) = \int_{-\infty}^{t} e^{-a(t-\tau)}\, x(\tau)\, d\tau$$

Impulse response: $h(t) = e^{-at} u(t)$

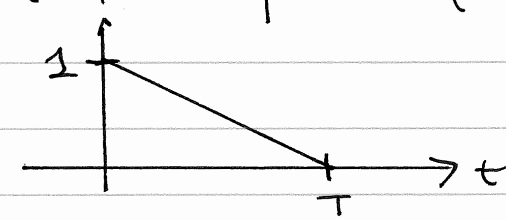
Area in the past exponentially weighted less



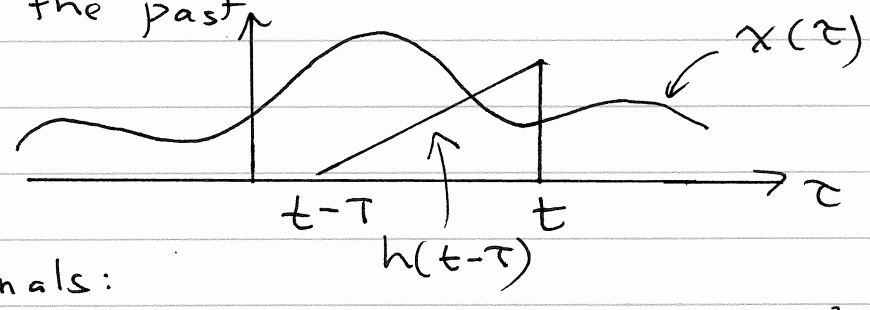
System 2:

$$y(t) = \frac{-1}{T} \int_{t-T}^{t} (t-\tau-T)\, x(\tau)\, d\tau$$

Impulse response: $h(t) = -\frac{1}{T}(t-T)\big(u(t)-u(t-T)\big)$



Area in the past linearly weighted less and cut-off at T secs in the past



Gaussian Signals:

$$\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{t^2}{2\sigma_1^2}} * \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{t^2}{2\sigma_2^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_3} e^{-\frac{t^2}{2\sigma_3^2}}$$

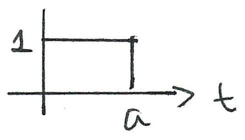
where: $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$

Basic CT Convolution Results

$$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}e^{at}u(t) + \frac{1}{b-a}e^{bt}u(t)$$

for $a \neq b$
includes $b=0$
for unit step

$$(u(t)-u(t-a)) * (u(t)-u(t-b))$$



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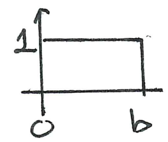

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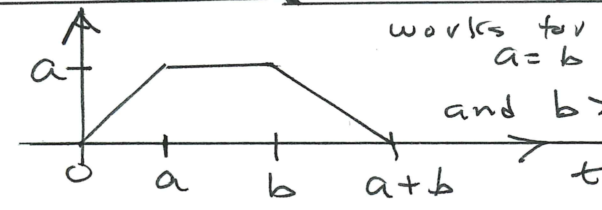
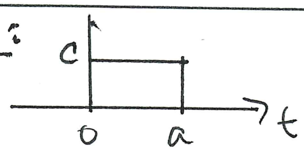


works for $a=b$
and $b>a$

If: $y(t) = x(t) * h(t)$, then: $a\,x(t-t_1) * b\,h(t-t_2) = ab\,y(t-(t_1+t_2))$

Thus:



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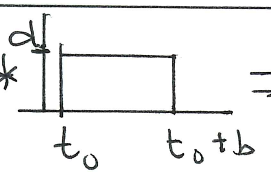

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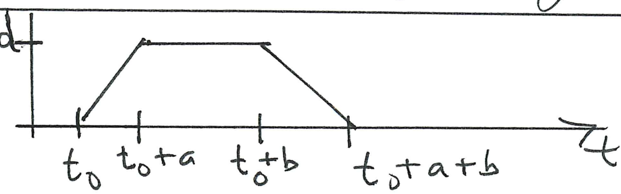


Variations on Decaying Exponentials Convolution

$$e^{at}u(t) * (u(t)-u(t-T)) = e^{at}u(t) * u(t) - e^{at}u(t) * u(t-T)$$

let $z(t) =$ formula at top of page with $b=0 = \frac{1}{a}(e^{at}-1)u(t) = z(t)$

Then: $e^{at}u(t) * (u(t)-u(t-T)) = z(t) - z(t-T)$

$$e^{at}(u(t)-u(t-T)) * u(t) = e^{at}u(t) * u(t) - e^{aT}e^{a(t-T)}u(t-T) * u(t)$$

$\boxed{z(t) = \frac{1}{a}(e^{at}-1)u(t)}$ $\Rightarrow$ $= z(t) - e^{aT}z(t-T)$

$$\underbrace{(at+b)}_{\text{turned-on forever}} * \underbrace{\text{rect}(t)}_{u(t+\frac{1}{2})-u(t-\frac{1}{2})} = at+b$$

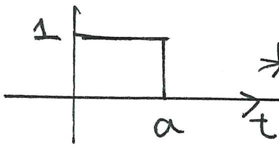
$$(at+b) * \underbrace{(u(t)-u(t-1))}_{\text{rect}(t-\frac{1}{2})}$$

$$= a(t-\tfrac{1}{2}) + b$$

Convolving with a Dirac-Delta Function

$x(t) * \delta(t) = x(t)$ $x(t) * \delta(t-t_0) = x(t-t_0)$

$$x(t) * (a\delta(t-t_1) + b\delta(t-t_2))$$
$$= a\,x(t-t_1) + b\,x(t-t_2)$$



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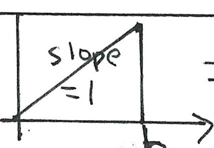

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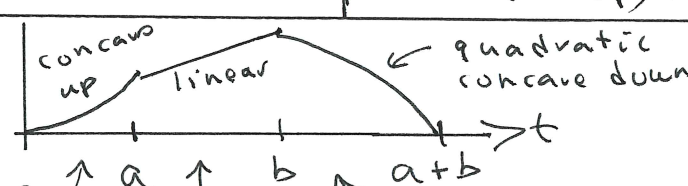
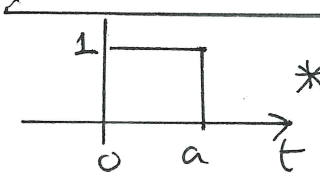


$b \geq a$

$\frac{t^2}{2}$ $at - \frac{a^2}{2}$ $-\frac{t^2}{2} + at + \frac{b^2-a^2}{2}$



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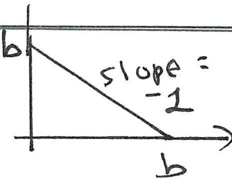

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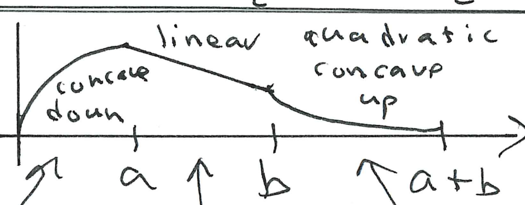


$-(t-b)$

$b \geq a$

$-\frac{t^2}{2} + bt$ $-at + \frac{a^2}{2} + ab$ $\frac{t^2}{2} - (a+b)t + \frac{(a+b)^2}{2}$

Impulse Response of LTI System

CT2

$$\delta(t) \rightarrow \boxed{\begin{matrix}\text{CT}\\ \text{LTI}\end{matrix}} \rightarrow h(t)$$

Properties determined from impulse response

causal: $h(t)=0$ for $t<0$

stable: if $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

memoryless: if $h(t)=K\delta(t)$

For any other input $x(t)$ output may be determined from convolution

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau=\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$y(t)=\int_{-\infty}^{t} e^{-2(t-\tau)}\,x(\tau)\,d\tau$$ Find output when: $x(t)=e^{-3t}u(t)$

Impulse Response: $h(t)=\int_{-\infty}^{t} e^{-2(t-\tau)}\,\delta(\tau)\,d\tau=\int_{-\infty}^{t} e^{-2t}\,\delta(\tau)\,d\tau=e^{-2t}\int_{-\infty}^{t}\delta(\tau)\,d\tau$

$$\boxed{\int_{-\infty}^{t}\delta(\tau)\,d\tau=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}=u(t)}$$ $h(t)=e^{-2t}u(t)$

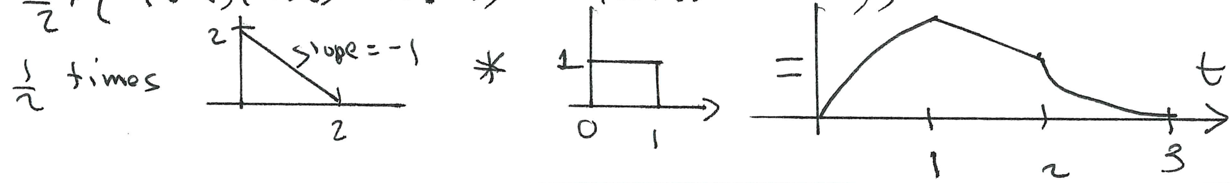
$$y(t)=e^{-2t}u(t)*e^{-3t}u(t)=\frac{1}{-2-(-3)}e^{-2t}u(t)+\frac{1}{-3-(-2)}e^{-3t}u(t)$$

$$y(t)=\int_{t-2}^{t} -\tfrac{1}{2}(t-\tau-2)\,x(\tau)\,d\tau$$ Find output when: $x(t)=u(t)-u(t-1)$

Impulse Response: $h(t)=\int_{t-2}^{t} -\tfrac{1}{2}(t-\tau-2)\,\delta(\tau)\,d\tau=-\tfrac{1}{2}(t-2)\int_{t-2}^{t}\delta(\tau)\,d\tau$

$$\boxed{\int_{t-2}^{t}\delta(\tau)\,d\tau=\begin{cases}1, & t>0 \text{ and } t-2<0\\ 0, & \text{otherwise}\end{cases}=u(t)-u(t-2)}=\frac{-1}{2}(t-2)\big(u(t)-u(t-2)\big)$$

Thus: $y(t)=\frac{1}{2}\times\{-(t-2)(u(t)-u(t-2)) * (u(t)-u(t-1))\}$



$$\delta[n] \rightarrow \boxed{\begin{matrix}\text{DT}\\ \text{LTI}\end{matrix}} \rightarrow h[n]$$

Properties determined from impulse response

causal: $h[n]=0$ for $n<0$

Stable: if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

memoryless: $h[n]=K\delta[n]$

For any other input, output may be determined from DT convolution

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]=\ldots x[-2]h[n+2]+x[-1]h[n+1]+x[0]h[n]$$
$$+\,x[1]h[n-1]+x[2]h[n-2]+\ldots$$

$= x[n]*h[n]$

leads to Table Method of Computing DT convolution

Basic DT Convolution Results: Geometric Sequences

$$\boxed{\alpha^n u[n] * \beta^n u[n] = \frac{\beta}{\beta-\alpha}\beta^n u[n] + \frac{\alpha}{\alpha-\beta}\alpha^n u[n]}$$ for $\alpha \neq \beta$, includes $\beta = 1$

Note: $y[n] = \alpha y[n-1] + x[n]$ Impulse response is $h[n] = \alpha^n u[n]$

Thus: if input is $x[n] = \beta^n u[n]$, then output is $y[n] = \beta^n u[n] * \alpha^n u[n]$ and one can use the formula above

Many variations: $\beta = 1$

$$\alpha^n u[n] * u[n] = \frac{1}{1-\alpha} u[n] + \frac{\alpha}{\alpha-1}\alpha^n u[n] = \frac{1}{1-\alpha}\{1-\alpha^{n+1}\}u[n]$$

Remember Time-Invariance and Linearity

If: $y[n] = x[n] * h[n]$, then: $a\,x[n-n_1] * b\,h[n-n_2] = ab\; y[n-(n_1+n_2)]$

Continuing variations: remember Distributive Property of Convolution

$$\alpha^n u[n] * \{u[n] - u[n-N]\} = \alpha^n u[n] * u[n] - \alpha^n u[n] * u[n-N]$$

$$= z[n] - z[n-N] \qquad z[n] = \frac{1}{1-\alpha}(1-\alpha^{n+1})u[n]$$

$\alpha^n\{u[n]-u[n-N_1]\} * \beta^n\{u[n]-u[n-N_2]\}$ ⇒ Use Foil Distributive Prop.

$$\underbrace{\alpha^n u[n] * \beta^n u[n]}_{z[n]} - \alpha^n u[n] * \beta^{N_2}\beta^{n-N_2}u[n-N_2]$$

$$- \alpha^{N_1}\alpha^{n-N_1}u[n-N_1] * \beta^n u[n]$$

$$+ \alpha^{N_1}\beta^{N_2}\alpha^{n-N_1}u[n-N_1] * \beta^{n-N_2}u[n-N_2] \quad \Big\}\text{ just shuts everything off at } N_1+N_2$$

$$= z[n] - \beta^{N_2} z[n-N_2] - \alpha^{N_1} z[n-N_1] \quad \text{for } 0 \le n \le N_1 + N_2 - 2$$

BUT if N_1 and N_2 are not large, use Table Method

Suppose $N_1 = 3$ and $N_2 = 5$: $x[n] = \{\underset{\uparrow n=0}{1}, \alpha, \alpha^2\}$ $h[n] = \{\underset{\uparrow n=0}{1}, \beta, \beta^2, \beta^3, \beta^4\}$

n	0	1	2	3	4	5	6	7
$1 \cdot h[n]$	1	β	β^2	β^3	β^4	0	0	length =
$\alpha \cdot h[n-1]$	0	α	$\alpha\beta$	$\alpha\beta^2$	$\alpha\beta^3$	$\alpha\beta^4$	0	$3+5-1=7$
$\alpha^2 h[n-2]$	0	0	α^2	$\alpha^2\beta$	$\alpha^2\beta^2$	$\alpha^2\beta^3$	$\alpha^2\beta^4$	
$y[n]$	1	$\alpha+\beta$	$\beta^2+\alpha\beta+\alpha^2$	$\beta^3+\alpha\beta^2+\alpha^2\beta$	$\beta^4+\alpha\beta^3+\alpha^2\beta^2$	$\alpha\beta^4+\alpha^2\beta^3$	$\alpha^2\beta^4$	$0 \le n \le 6$

On exam, just use specific values of α and β above

Convolving 2 DT rectangles:

$$(u[n] - u[n-(N_1+1)]) * (u[n] - u[n-(N_2+1)]) = \{\underset{\uparrow n=0}{1}, 2, \ldots, N_1, \overbrace{N_1+1, \ldots, N_1+1}^{(N_2+1)-(N_1+1)+1 \text{ times at peak value}}, N_1, \ldots, 2, 1\}$$

length $= (N_1+1) + (N_2+1) - 1$

Can always use Table Method as alternative to convolve 2 DT rectangles

$x[n] = \{\underset{\uparrow n=0}{1}, 1, 1\} = u[n] - u[n-3]$ $\quad h[n] = \{\underset{\uparrow n=0}{1}, 1, 1, 1, 1\} = u[n] - u[n-5]$

n	0	1	2	3	4	5	6
$x[0]h[n]$	1	1	1	1	1	0	0
$x[1]h[n-1]$	0	1	1	1	1	1	0
$x[2]h[n-2]$	0	0	1	1	1	1	1
$y[n]$	1	2	3	3	3	2	1

length $= 3+5-1 = 8-1 = 7$

$0 \leq n \leq 6$

peak value $= 3 =$ length of narrower rectangle

$y[n] = \{\underset{\uparrow n=0}{1}, 2, 3, 3, 3, 2, 1\}$

no. of times at peak value $= 5-3+1 = 3$ no. of times at peak value

Difference Eqn. $y[n] = a\,y[n-1] + x[n] - a^D x[n-D]$ $\quad D > 1$ integer

Impulse Response $h[n] = a^n\{u[n] - u[n-D]\}$

Includes $a=1$ as special case:

$y[n] = y[n-1] + x[n] - x[n-D] \Rightarrow h[n] = u[n] - u[n-D]$

Example: $y[n] = y[n-1] + x[n] - x[n-5]$ $\quad x[n] = u[n] - u[n-3] = \{\underset{\uparrow n=0}{1}, 1, 1\}$

$h[n] = u[n] - u[n-5] = \{\underset{\uparrow n=0}{1}, 1, 1, 1, 1\}$

output: $y[n] = x[n] * h[n]$ is at top of this page

Another Example: $y[n] = \beta\, y[n-1] + x[n] - \beta^5 x[n-5]$ $\quad$ input is: $x[n] = \alpha^n\{u[n] - u[n-3]\}$

$h[n] = \beta^n\{u[n] - u[n-5]\}$ $\quad = \{\underset{\uparrow n=0}{1}, \alpha, \alpha^2\}$

output $y[n] = x[n] * h[n]$ computed near bottom of previous page

$y[n] = \{\underset{\uparrow n=0}{1}, \alpha+\beta, \beta^2+\alpha\beta+\alpha^2, \beta^3+\alpha\beta^2+\alpha^2\beta, \beta^4+\alpha\beta^3+\alpha^2\beta^2, \alpha\beta^4+\alpha^2\beta^3, \alpha^2\beta^4\}$

If the Difference Eqn. is non-recursive, i.e., does not depend on past value of $y[n]$ i.e., does not depend on $y[n-1]$, $y[n-2]$

$$y[n] = \sum_{k=0}^{M} b_k\, x[n-k] \Rightarrow h[n] = \sum_{k=0}^{M} b_k\, \delta[n-k] = \{\underset{\uparrow n=0}{b_0}, b_1, \ldots, b_M\}$$

Example:

$y[n] = x[n] + 3x[n-1] - 5x[n-2] + 2x[n-3]$

$h[n] = \{\underset{\uparrow n=0}{1}, 3, -5, 2\}$

$x[n] * \delta[n] = x[n]$ $\quad x[n] * \delta[n-n_0] = x[n-n_0]$

Convolving with a Kronecker Delta Function.

Eg. $x[n] * \{a\delta[n] + b\delta[n-1] + c\delta[n-2]\} = a\,x[n] + b\,x[n-1] + c\,x[n-2]$