Proof that DT System Described by Difference Equation is Linear and TI

$$x_1[n] \longrightarrow \boxed{S} \longrightarrow y_1[n]$$

$$y_1[n] = -\sum_{k=1}^{N} a_k \, y_1[n-k] + \sum_{k=0}^{M} b_k \, x_1[n-k] \qquad (1)$$

$$x_2[n] \longrightarrow \boxed{S} \longrightarrow y_2[n]$$

$$y_2[n] = -\sum_{k=1}^{N} a_k \, y_2[n-k] + \sum_{k=0}^{M} b_k \, x_2[n-k] \qquad (2)$$

Proof invokes distributive property of multiplication

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Consider α_1 1 $+ \alpha_2$ 2 $=$

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] =$$

$$-\alpha_1 \sum_{k=1}^{N} a_k y_1[n-k] - \alpha_2 \sum_{k=1}^{N} a_k y_2[n-k]$$

$$+\alpha_1 \sum_{k=0}^{M} b_k x_1[n-k] + \alpha_2 \sum_{k=0}^{M} b_k x_2[n-k]$$

$$= -\sum_{k=1}^{N} a_k \left\{ \alpha_1 y_1[n-k] + \alpha_2 y_2[n-k] \right\}$$

$$+\sum_{k=0}^{M} b_k \left\{ \alpha_1 x_1[n-k] + \alpha_2 x_2[n-k] \right\} \quad (3)$$

If the system is linear, then that's exactly the ean we obtain if we input $\alpha_1 x_1[n] + \alpha_2 x_2[n]$ into the system

$$x[n] \longrightarrow \boxed{S} \longrightarrow y[n]$$

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. If S is linear, then:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{S} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

$$\left.\begin{aligned} &\alpha_1 y_1[n] + \alpha_2 y_2[n] = \\ &= -\sum_{k=1}^{N} a_k \left\{ \alpha_1 y_1[n-k] + \alpha_2 y_2[n-k] \right\} \\ &+ \sum_{k=0}^{M} b_k \left\{ \alpha_1 x_1[n-k] + \alpha_2 x_2[n-k] \right\} \end{aligned}\right\}$$

Proved this was true on previous page $\Rightarrow$ System is Linear!

<u>If</u>: $y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$

<u>Then</u>: $y[n-n_0] = -\sum_{k=1}^{N} a_k\, y[n-n_0-k] + \sum_{k=0}^{M} b_k\, x[n-n_0-k]$ (A)

If the system is TI, then:

$$x[n-n_0] \longrightarrow \boxed{S} \longrightarrow y[n-n_0]$$

such that this input-output satisfies the diff. eqn.

$y[n-n_0] = -\sum_{k=1}^{N} a_k\, y[n-k-n_0] + \sum_{k=0}^{M} b_k\, x[n-k-n_0]$ (B)

Due to the commutativity of addition, eqns (A) and (B) are the same

$\Rightarrow$ System is Time-Invariant (TI)

- Difference Eqns. with constant coefficients are LTI Systems (DT)
- Difference Eqns are used all the time in practice for a variety of applications
- They can be used as frequency selective filters (lowpass, band pass, highpass), notch filters, and many other useful functions
- Since they are LTI, they are completely characterized by their impulse response as we will see in Chap. 2

Proof that DT System Described by Difference Equation is Linear and TI

$$x_1[n] \longrightarrow \boxed{S} \longrightarrow y_1[n]$$

$$y_1[n] = -\sum_{k=1}^{N} a_k\, y_1[n-k] + \sum_{k=\boxed{-M}}^{M} b_k\, x_1[n-k] \qquad (1)$$

if you change the lower limit as shown here, the proof that the system is LTI is exactly the same. The only thing is that the system is not causal.

$$x_2[n] \longrightarrow \boxed{S} \longrightarrow y_2[n]$$

$$y_2[n] = -\sum_{k=1}^{N} a_k\, y_2[n-k] + \sum_{k=\boxed{-M}}^{M} b_k\, x_2[n-k] \qquad (2)$$

Proof invokes distributive property of multiplication

Example

Hmwk. Prob. 1.18

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

- This is a difference eqn. It's just non-recursive ($a_k = 0$) and non-causal } see p. 7a
- So it's LTI. But we'll prove it anyhow

(a) Linear? Inputting $x_1[n]$ and $x_2[n]$ individually we have:

$$① \; y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k] \qquad ② \; y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

α_1 ① $+ \alpha_2$ ② $\Rightarrow$

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1 \sum_{k=n-n_0}^{n+n_0} x_1[k] + \alpha_2 \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

Rearranging:

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \sum_{k=n-n_0}^{n+n_0} \left\{ \alpha_1 x_1[k] + \alpha_2 x_2[k] \right\}$$

If the system is linear, that's exactly the equation we would obtain if we input $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ into the system

i.e., if S is linear:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{S} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

• which, in turn, dictates:

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \sum_{k=n-n_0}^{n+n_0} \left(\alpha_1 x_1[k] + \alpha_2 x_2[k] \right)$$

and we already proved that this eqn. holds

$\Rightarrow$ system is linear

Side-note: $y[n]=\sum_{k=n-n_0}^{n+n_0} x[k]$

- change of variables: $k'=-n+k$

$k'=k-n$

new limits on sum:

$k\Big]_{n-n_0}^{n+n_0} \Rightarrow k'=\Big]_{n-n_0-n=-n_0}^{n+n_0-n=n_0}$

- **Substitute:** $k=k'+n$

$y[n]=\sum_{k'=-n_0}^{n_0} x[n+k']$ } This is a non-causal difference eqn.

You could do another change of variables $k=-k'$ to obtain $y[n]=\sum_{k=-n_0}^{n_0} x[n-k]$

• **Is System TI?** First, note: since n_0 is used in system equation, use ℓ for time-shift in input signal:

$$x[n-\ell] \longrightarrow \boxed{S} \longrightarrow z[n]$$

$$\boxed{\text{Is } z[n] = y[n-\ell] ?}$$

$$z[n] = \sum_{k=n-n_0}^{n+n_0} x[k-\ell]$$

change of variables: $k' = k - \ell$

new limits:

$$k \Big]_{n-n_0}^{n+n_0} \Rightarrow k' \Big]_{n-n_0-\ell}^{n+n_0-\ell}$$

substitute: $\boxed{k = k' + \ell}$

$$z[n] = \sum_{k'=n-\ell-n_0}^{n-\ell+n_0} x[k']$$

Recall:

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k] \implies y[n-\ell] = \sum_{k=n-\ell-n_0}^{n-\ell+n_0} x[k]$$

Thus:

Thus: $z[n] = y[n-\ell] \Rightarrow$ system is TI

Part (c): Is system stable? Yes

- as long as $x[n] < B$ for all n, where $B < \infty$
- Then the "worst case" or largest value $y[n]$ can be if $x[k] = B$ within the window $n-n_0 \leq k \leq n+n_0$
- Thus, $y[n] < (2n_0+1)B$

$\Rightarrow$ system is stable