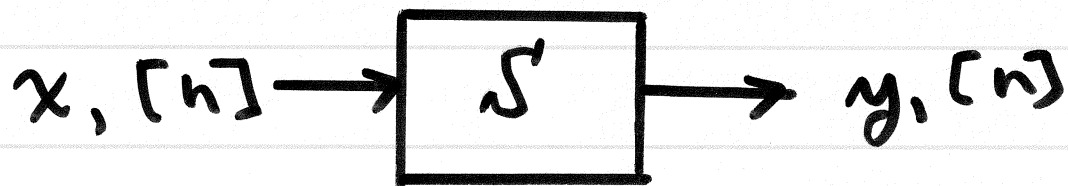
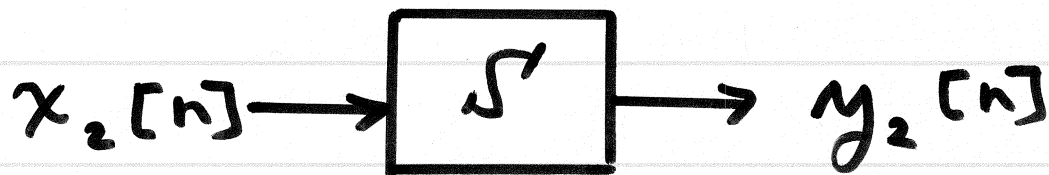


Proof that DT System Described by Difference Equation is Linear and TI ①



$$y_1[n] = -\sum_{k=1}^N a_k y_1[n-k] + \sum_{k=0}^M b_k x_1[n-k] \quad \text{①}$$



$$y_2[n] = -\sum_{k=1}^N a_k y_2[n-k] + \sum_{k=0}^M b_k x_2[n-k] \quad \text{②}$$

Proof invokes distributive property of multiplication

Consider α_1 (1) + α_2 (2) = (2)

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] =$$

$$- \alpha_1 \sum_{k=1}^N a_k y_1[n-k] - \alpha_2 \sum_{k=1}^N a_k y_2[n-k]$$

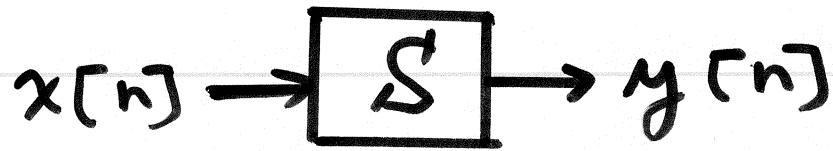
$$+ \alpha_1 \sum_{k=0}^M b_k x_1[n-k] + \alpha_2 \sum_{k=0}^M b_k x_2[n-k]$$

$$= - \sum_{k=1}^N a_k \{ \alpha_1 y_1[n-k] + \alpha_2 y_2[n-k] \}$$

$$+ \sum_{k=0}^M b_k \{ \alpha_1 x_1[n-k] + \alpha_2 x_2[n-k] \}$$

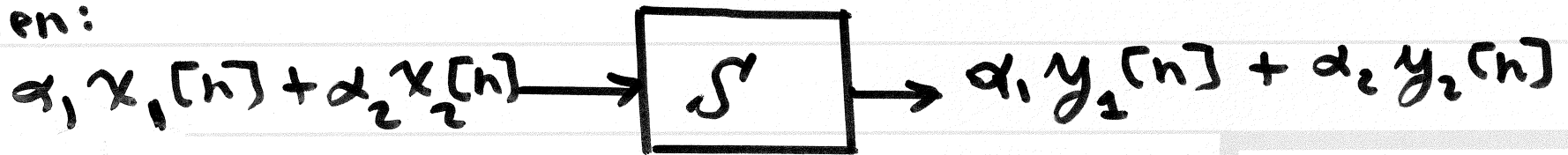
(3)

If the system is linear, then that's exactly the way we obtain it we input $\alpha_1 x_1[n] + \alpha_2 x_2[n]$ into the system ③



$$y[n] = - \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k]$$

Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. If S is linear, then:



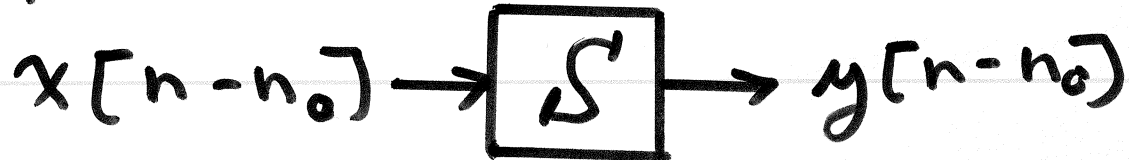
$$\begin{aligned} & \alpha_1 y_1[n] + \alpha_2 y_2[n] = \\ & = - \sum_{k=1}^N a_k \{ \alpha_1 y_1[n-k] + \alpha_2 y_2[n-k] \} \\ & + \sum_{k=0}^M b_k \{ \alpha_1 x_1[n-k] + \alpha_2 x_2[n-k] \} \end{aligned}$$

Proved this was true on previous page \Rightarrow System is Linear!

$$\underline{\text{If:}} \quad y[n] = -\sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k] \quad (4)$$

$$\underline{\text{Then:}} \quad y[n-n_0] = -\sum_{k=1}^N a_k y[n-n_0-k] + \sum_{k=0}^M b_k x[n-n_0-k] \quad (A)$$

If the system is TI, then:



such that this input-output satisfies the diff. eqn.

$$y[n-n_0] = -\sum_{k=1}^N a_k y[n-k-n_0] + \sum_{k=0}^M b_k x[n-k-n_0] \quad (B)$$

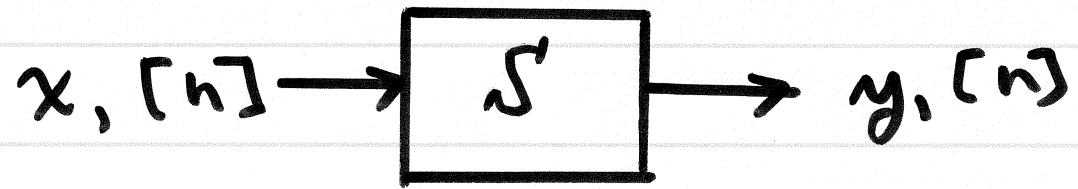
Due to the commutativity of addition,

eqns (A) and (B) are the same

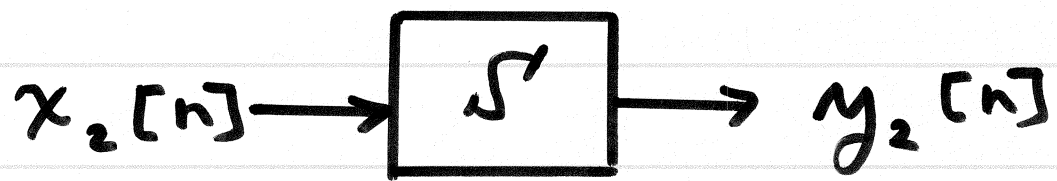
\Rightarrow System is Time-Invariant (TI)

- ⑤
- Difference Eqns. with constant coefficients are LTI Systems (DT)
 - Difference Eqns are used all the time in practice for a variety of applications
 - They can be used as frequency selective filters (low pass, band pass, high pass), notch filters, and many other useful functions
 - Since they are LTI, they are completely characterized by their impulse response as we will see in Chap. 2

Proof that DT System Described by Difference Equation is Linear and TI ①



$$y_1[n] = - \sum_{k=1}^N a_k y_1[n-k] + \sum_{k=-M}^M b_k x_1[n-k] \quad \text{①}$$



$$y_2[n] = - \sum_{k=1}^N a_k y_2[n-k] + \sum_{k=-M}^M b_k x_2[n-k] \quad \text{②}$$

if you change the lower limit as shown here, the proof that the system is LTI is exactly the same. The only thing is that the system is not causal.

Proof invokes distributive property of multiplication

Example

Hmwk. Prob. 1.18

(6)

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

• This is a difference eqn. It's just non-recursive ($a_k=0$) and non-causal } see pg (7A)

• So it's LTI. But we'll prove it anyhow

(a) Linear? Inputting $x_1[n]$ and $x_2[n]$ individually we have:

$$\textcircled{1} y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

$$\textcircled{2} y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

$$\alpha_1 \textcircled{1} + \alpha_2 \textcircled{2} \Rightarrow$$

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1 \sum_{k=n-n_0}^{n+n_0} x_1[k] + \alpha_2 \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

Rearranging:

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \sum_{k=n-n_0}^{n+n_0} \left\{ \alpha_1 x_1[k] + \alpha_2 x_2[k] \right\} \quad (7)$$

If the system is linear, that's exactly the equation we would obtain if we input

$x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ into the system

i.e., if S is linear:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \rightarrow \boxed{S} \rightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

• which, in turn, dictates:

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \sum_{k=n-n_0}^{n+n_0} \left(\alpha_1 x_1[k] + \alpha_2 x_2[k] \right)$$

and we already proved that this eqn. holds

\Rightarrow system is linear

7a

Side-note: $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$

• change of variables: $k' = -n + k$
 $k' = k - n$

new limits on sum:

$$k \Big]_{n-n_0}^{n+n_0} \Rightarrow k' \Big]_{n-n_0-n=-n_0}^{n+n_0-n=n_0}$$

• Substitute: $k = k' + n$

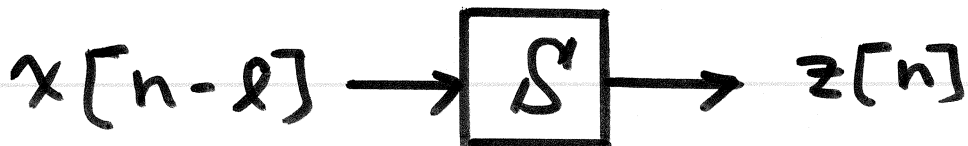
$$y[n] = \sum_{k'=-n_0}^{n_0} x[n+k']$$

This is a non-causal difference eqn.

You could do another change of variables $k = -k'$

to obtain $y[n] = \sum_{k=-n_0}^{n_0} x[n-k]$

• Is System TI? First, note: since n_0 is used in system equation, use l for time-shift in input signal:



Is $z[n] = y[n-l]$?

$$z[n] = \sum_{k=n-n_0}^{n+n_0} x[k-l]$$

change of variables: $k' = k - l$

new limits:

$$\left. \begin{array}{l} k \\ \hline \end{array} \right]_{n-n_0}^{n+n_0} \Rightarrow \left. \begin{array}{l} k' \\ \hline \end{array} \right]_{n-n_0-l}^{n+n_0-l}$$

substitute: $\boxed{k = k' + l}$

$$z[n] = \sum_{k=n-l-n_0}^{n-l+n_0} x[k']$$

Recall:

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

Thus:

$$y[n-l] = \sum_{k=n-l-n_0}^{n-l+n_0} x[k]$$

⑨

Thus: $z[n] = y[n-l] \Rightarrow$ system is TI

Part (c): Is system stable? Yes

• as long as $x[n] < B$ for all n , where $B < \infty$

• Then the "worst case" or largest value $y[n]$ can be if $x[k] = B$ within the window $n-n_0 \leq k \leq n+n_0$

• Thus, $y[n] < (2n_0+1)B$

\Rightarrow system is stable