Discrete-Time Convolution Examples

without explicitly doing convolution, but rather using basic convolution results plus properties of convolution

- Basic Convolution Results

- Example 2.3 in text

$$\left.\begin{aligned} x[n] &= \alpha^n u[n] \\ h[n] &= u[n] \end{aligned}\right\} y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \frac{1}{1-\alpha}\left(1-\alpha^{n+1}\right)u[n]$$

$$= \frac{1}{1-\alpha}u[n] - \frac{\alpha}{1-\alpha}\alpha^n u[n]$$

- Special case of Prob. 2.21 (a) which was worked out in a previous set of handout notes

- $\left.\begin{aligned} x[n] &= \alpha^n u[n] \\ h[n] &= \beta^n u[n] \end{aligned}\right\} y[n] = x[n] * h[n] =$

$$= c_1 \beta^n u[n] + c_2 \alpha^n u[n]$$

where: $c_1 = \dfrac{\beta}{\beta - \alpha}$ $\quad c_2 = \dfrac{-\alpha}{\beta - \alpha}$

- Ex. 2.3 in text is special case where $\beta = 1$

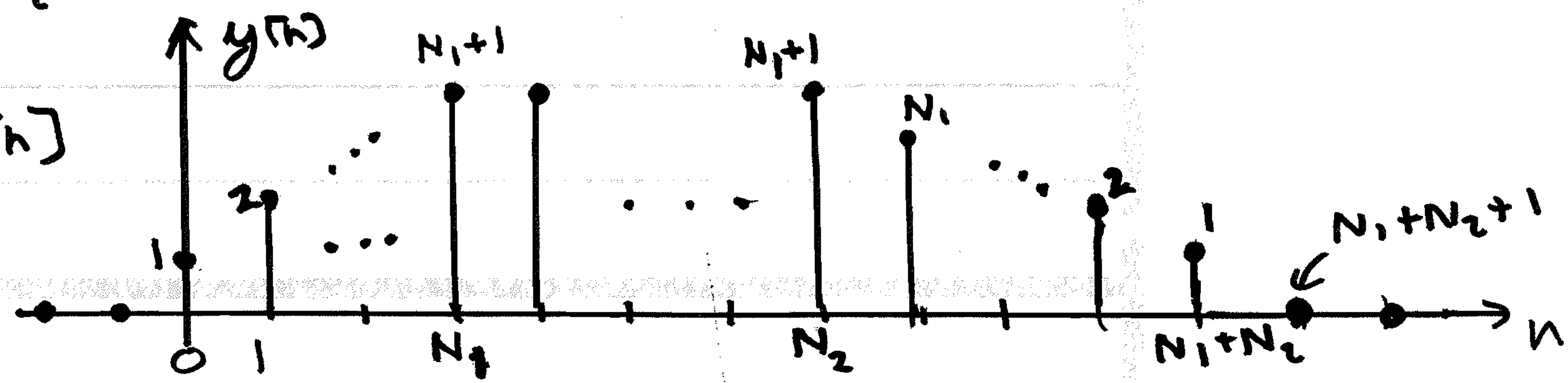
- <u>Other basic result:</u> convolution of two DT rectangles

$x[n] = u[n] - u[n-(N_1+1)] = \begin{cases} 1, & 0 \le n \le N_1 \\ 0, & \text{otherwise} \end{cases}$ "length" $= N_1+1$

$h[n] = u[n] - u[n-(N_2+1)] = \begin{cases} 1, & 0 \le n \le N_2 \\ 0, & \text{otherwise} \end{cases}$ "length" $= N_2+1$

assume $N_2 \ge N_1$ WLOG

$y[n] =$
$x[n] * h[n]$



- <u>Properties of convolution:</u>

(i) associativity, (ii) commutativity, (iii) distributive

- Recall and use concepts of <u>time-invariance</u> and <u>linearity</u> (homogeneity and superposition)

- Leads to following results:

- If $y[n] = x[n] * h[n]$, then:

1. $z[n] = x[n-n_0] * h[n] = y[n-n_0]$

2. $z[n] = x[n] * h[n-n_0] = y[n-n_0]$

3. $z[n] = x[n-n_1] * h[n-n_2] = y[n-(n_1+n_2)]$

n_1, n_2 can be either positive or negative

- Consider convolving two finite-length geometric sequences

$$x[n] = \alpha^n \{u[n] - u[n-N_1]\}$$

$$h[n] = \beta^n \{u[n] - u[n-N_2]\}$$

$$y[n] = x[n] * h[n] = ?$$

- We can use the properties of convolution to avoid performing convolution explicitly

$$x[n] * h[n] = \{\alpha^n u[n] - \alpha^{N_1} \alpha^{n-N_1} u[n-N_1]\} * \{\beta^n u[n] - \beta^{N_2} \beta^{n-N_2} u[n-N_2]\}$$

$$= \left\{ \begin{array}{l} \alpha^n u[n] * \beta^n u(n) \\ - \alpha^n u[n] * \beta^{N_2} \beta^{n-N_2} u[n-N_2] \\ - \alpha^{N_1} \alpha^{n-N_1} u[n-N_1] * \beta^n u(n) \\ + \alpha^{N_1} \beta^{N_2} \alpha^{n-N_1} u[n-N_1] * \beta^{n-N_2} u[n-N_2] \end{array} \right.$$

- again, define $c_1 = \frac{\beta}{\beta - \alpha}$ $\quad c_2 = \frac{-\alpha}{\beta - \alpha} = \frac{\alpha}{\alpha - \beta}$

- Final answer is: $y[n] = x[n] * h[n]$

$$
= \begin{cases}
c_1 \beta^n u[n] + c_2 \alpha^n u[n] \\
- \beta^{N_2} \left(c_1 \beta^{n-N_2} u[n-N_2] + c_2 \alpha^{n-N_2} u[n-N_2] \right) \\
- \alpha^{N_1} \left(c_1 \beta^{n-N_1} u[n-N_1] + c_2 \alpha^{n-N_1} u[n-N_1] \right) \\
+ \alpha^{N_1} \beta^{N_2} \left(c_1 \beta^{n-(N_1+N_2)} u[n-(N_1+N_2)] + c_2 \alpha^{n-(N_1+N_2)} u[n-(N_1+N_2)] \right)
\end{cases}
$$

- We obtained a closed-form answer without having to explicitly compute a convolution

- answer is general: α and β are arbitrary complex-valued (in general) constants, N_1, N_2 are positive integers

Example. Prob. 2.26 $\quad y[n] = x_1[n] * x_2[n] * x_3[n]$

$$x_1[n] = \left(\tfrac{1}{2}\right)^n u[n] \qquad x_2[n] = u[n+3] \qquad x_3[n] = \delta[n] - \delta[n-1]$$

(a) $x_1[n] * x_2[n] = ? = z[n]$

$\alpha = \frac{1}{2} \quad \beta = 1$ $\quad$ in basic result on pg 2 of this notes for DT convolution of two geometric sequences:

$$r_1 = \frac{\beta}{\beta - \alpha} = \frac{1}{1-\frac{1}{2}} = 2 \qquad r_2 = \frac{\alpha}{\alpha - \beta} = \frac{1/2}{1/2 - 1} = -1$$

$$\underbrace{x_1[n] * x_2[n]}_{} = 2\, u[n+3] - \left(\tfrac{1}{2}\right)^{n+3} u[n+3]$$

(b) $z[n] * x_3[n] \quad \Rightarrow$ easiest way

$$z[n] * x_3[n] = z[n] * \{\delta[n] - \delta[n-1]\}$$

$$= z[n] - z[n-1]$$

$$= 2\, u[n+3] - \left(\tfrac{1}{2}\right)^{n+3} u[n+3] - 2\, u[n+2] + \left(\tfrac{1}{2}\right)^{n+2} u[n+2]$$

$$= \delta[n+3] + \left(-\tfrac{1}{8} + \tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^n u[n+2]$$

$$= \delta[n+3] + \tfrac{1}{8}\left(\tfrac{1}{2}\right)^n u[n+2]$$

You do parts (c) and (d)

Prob. 2.26 (cont.)

Ans to (b) further simplifies as

$$z[n] = \frac{1}{8}\left(\frac{1}{2}\right)^n u[n+3] = \left(\frac{1}{2}\right)^{n+3} u[n+3]$$

Now (c) and (d):

First convolve $x_2[n]$ and $x_3[n]$

$$u[n+3] * (\delta[n] - \delta[n-1])$$

$$= u[n+3] * \delta[n] - u[n+3] * \delta[n-1]$$

$$= u[n+3] - u[n+2] = \delta[n+3]$$

Next, convolve this with $x_1[n]$

$$\left(\frac{1}{2}\right)^n u[n] * \delta[n+3] = \left(\frac{1}{2}\right)^{n+3} u[n+3]$$

- Demonstrates associativity of DT convolution:

$$(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$$

Example. Prob. 2.5 $\quad y[n] = x[n] * h[n]$

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-10]$$

$$h[n] = \begin{cases} 1, & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-(N+1)]$$

Find N given $y[4] = 5 \quad y[14] = 0$

From convolution of two DT rectangles result on bottom of pg. 2:

length of $y[n] = 10 + (N+1) - 1 = N + 10$

is nonzero from $n = 0$ to $n = N + 10 - 1 = N + 9$

So: $N + 9 < 14$ so $N < 5$

So $N_1 = N$ and $N_2 = 9$ at bottom of pg. 2

Check $N = 4 = N_1$, $\Rightarrow y[N_1] = N_1 + 1 \Rightarrow y[4] = 5$ ✓ checks

Answer: $N = 4$

Prob. 2.21 part (d) $y[n] = x[n] * h[n]$

$$x[n] = u[n] - u[n-5] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Define:

$$\tilde{h}[n] = u[n] - u[n-6] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

Then: $h[n] = \tilde{h}[n-2] + \tilde{h}[n-11]$

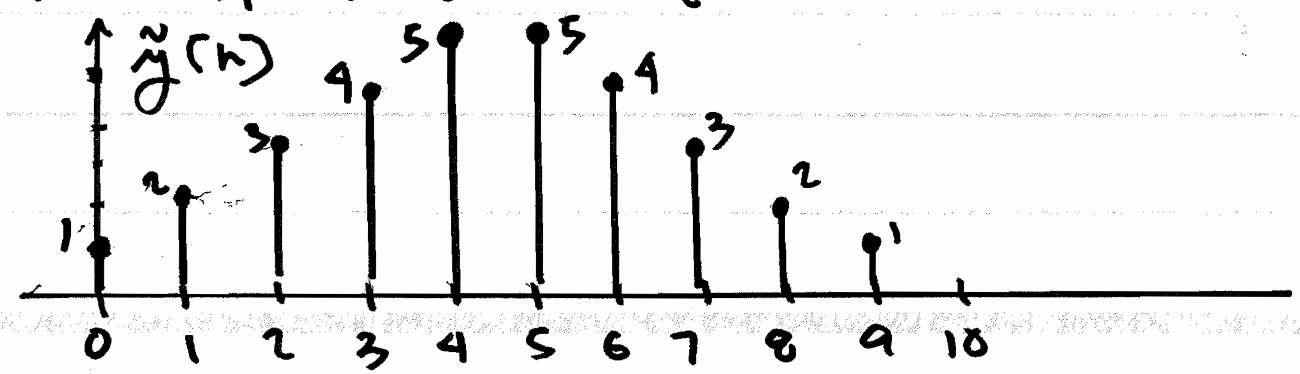
It follows: define $\tilde{y}[n] = x[n] * \tilde{h}[n]$

Then:

$$y[n] = \tilde{y}[n-2] + \tilde{y}[n-11]$$

where: $\tilde{y}[n]$ determined from bottom of pg. 2

with $N_1 = 4$ and $N_2 = 5$



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$$\tilde{y}[n] = \{1, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$$

$\uparrow$ $n=0$

$$y[n] = \tilde{y}(n-2) + \tilde{y}[n-11]$$

n:	0	1	2	3	4	5	6	7	8	9	10	11

$$= \{0, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$$

$$+ \{ \qquad 1, 2, 3, 4, 5, 5, 4, 3, 2, 1$$

Final answer:

$$y[n] = \{0, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 2, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$$

$\uparrow$ $n=0$

single point where the two terms overlap at $n=11$