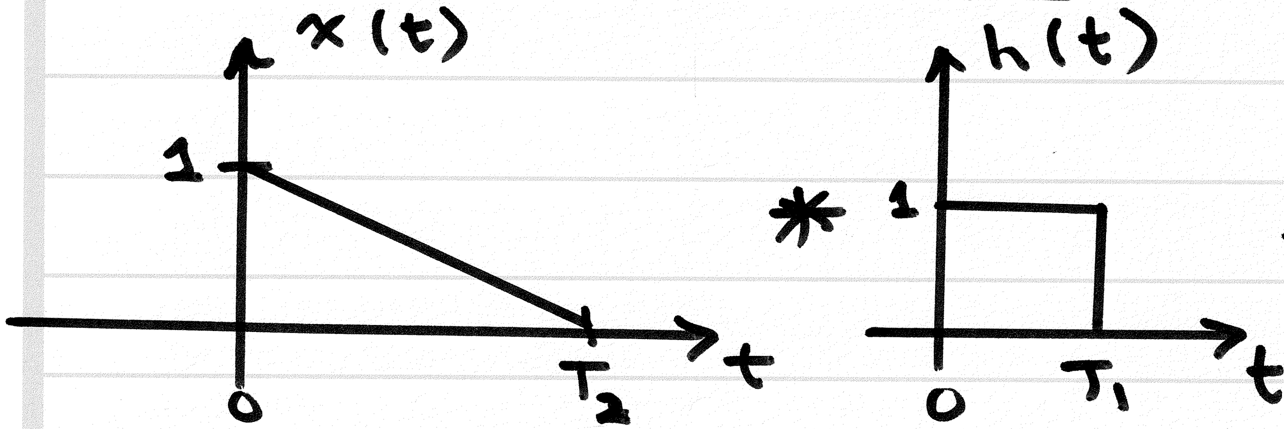
Convolution Example:



$= y(t) = ?$

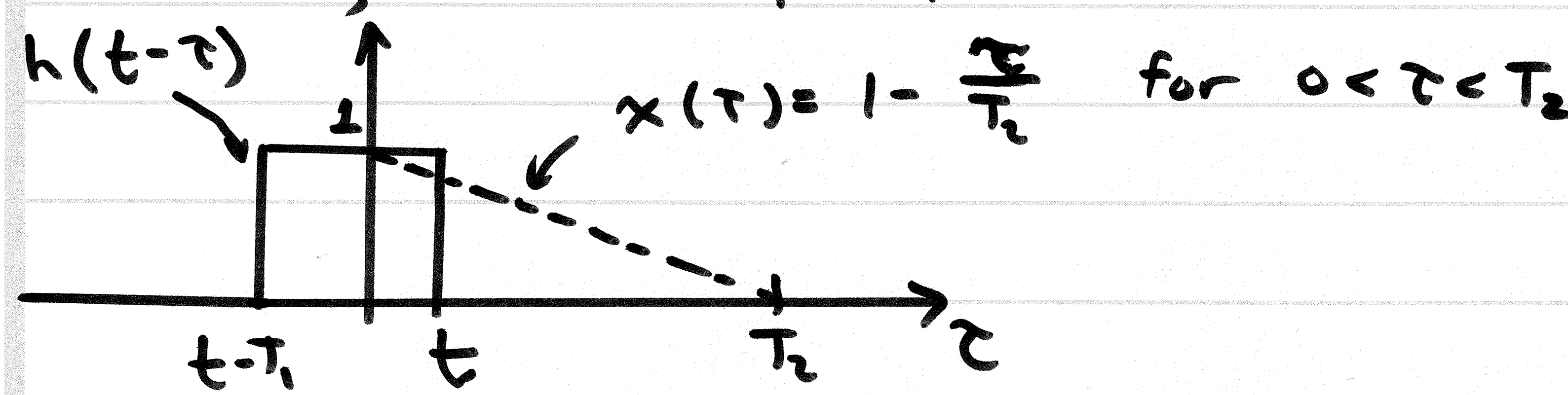
$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(t) = \mathrm{rect}\left(\frac{t - T_1/2}{T_1}\right)\end{array}} \longrightarrow y(t)$$

$$y(t) = \int_{t-T_1}^{t} x(\tau)\, d\tau$$

Assume $T_2 > T_1$

For $t < 0$, $y(t) = 0$ (no overlap)

• for $t>0$ and $t-T_1<0$ $(t<T_1)$,
that is, $0<t<T_1$ (partial overlap):



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t \left(1-\frac{\tau}{T_2}\right) d\tau = t\Big|_0^t - \frac{1}{T_2}\frac{\tau^2}{2}\Big]_0^t$$

$$= (t-0) - \frac{1}{2T_2}(t^2-0^2)$$

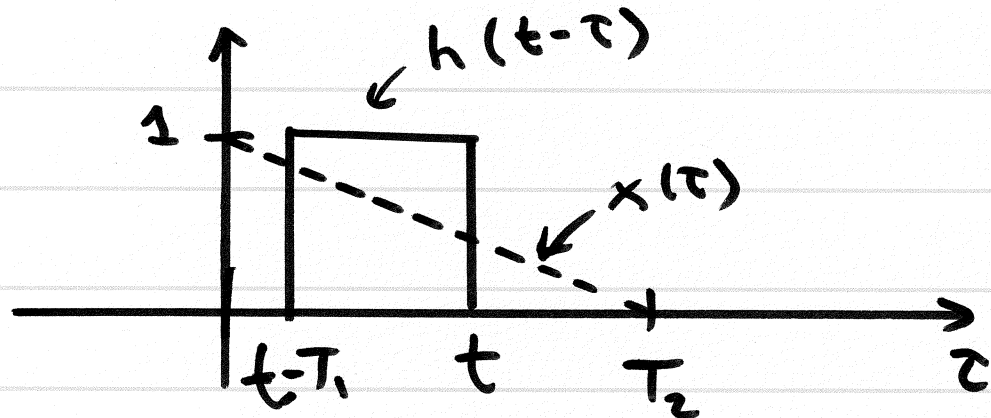
$$= -\frac{1}{2T_2}t^2 + t \qquad \text{for } 0<t<T_1$$

concave downwards

③

- for $t - T_1 > 0$ and $t < T_2$

$\Rightarrow \quad T_1 < t < T_2$ (full overlap):



$$y(t) = \int_{t-T_1}^{t} \left(1 - \frac{\tau}{T_2}\right) d\tau = \tau \Big]_{t-T_1}^{t} - \frac{1}{2T_2} \tau^2 \Big]_{t-T_1}^{t}$$

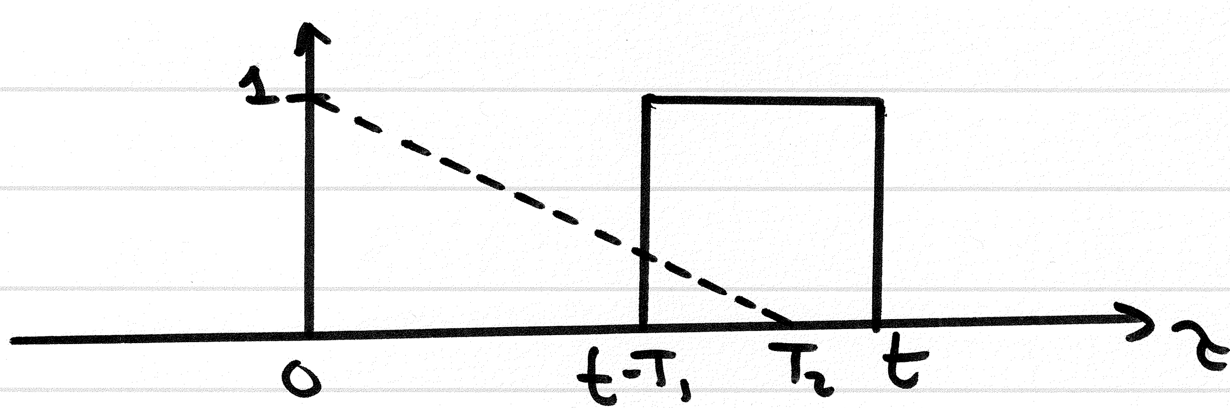
$$= \left[t - (t - T_1)\right] - \frac{1}{2T_2}\left\{t^2 - (t - T_1)^2\right\}$$

$$= T_1 - \frac{1}{2T_2}\left\{t^2 - t^2 + 2T_1 t - T_1^2\right\}$$

$$= -\frac{T_1}{T_2} t + \left(T_1 + \frac{T_1^2}{2T_2}\right)$$

• for $t > T_2$ and $t - T_1 < T_2$

$\Rightarrow$ $T_2 < t < T_1 + T_2$ (partial overlap):



$$y(t) = \int_{t-T_1}^{T_2} \left(1 - \frac{\tau}{T_2}\right) d\tau = \tau\Big]_{t-T_1}^{T_2} - \frac{1}{2T_2}\tau^2\Big]_{t-T_1}^{T_2}$$

$$= \left[T_2 - (t - T_1)\right] - \frac{1}{2T_2}\left\{T_2^2 - (t - T_1)^2\right\}$$

$$= T_1 + T_2 - t - \frac{T_1}{T_2}t + \frac{t^2}{2T_2} - \left(\frac{T_2^2 - T_1^2}{2T_2}\right)$$

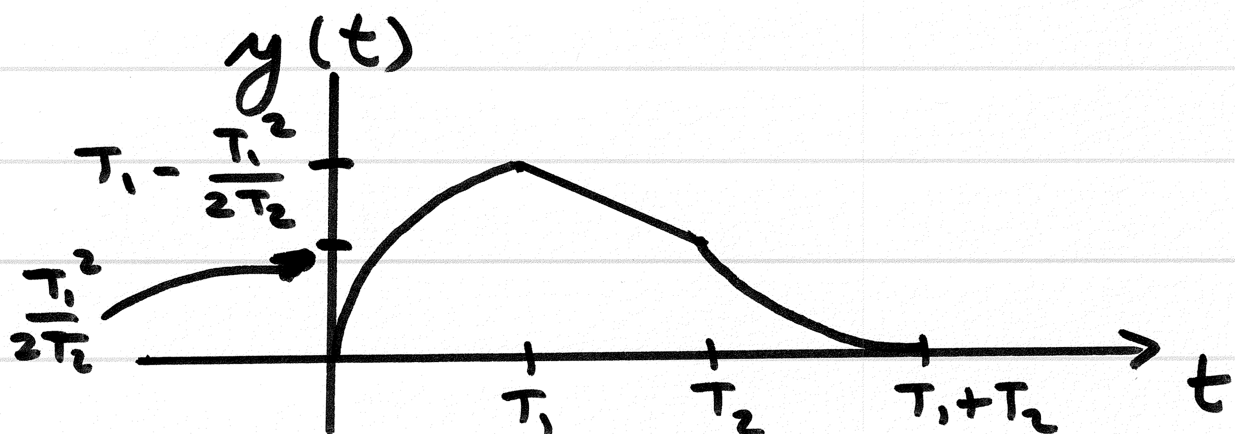
$$= +\frac{t^2}{2T_2} - \left(1 + \frac{T_1}{T_2}\right)t + \left\{T_1 + T_2 - \frac{T_2^2 - T_1^2}{2T_2}\right\}$$

concave upwards

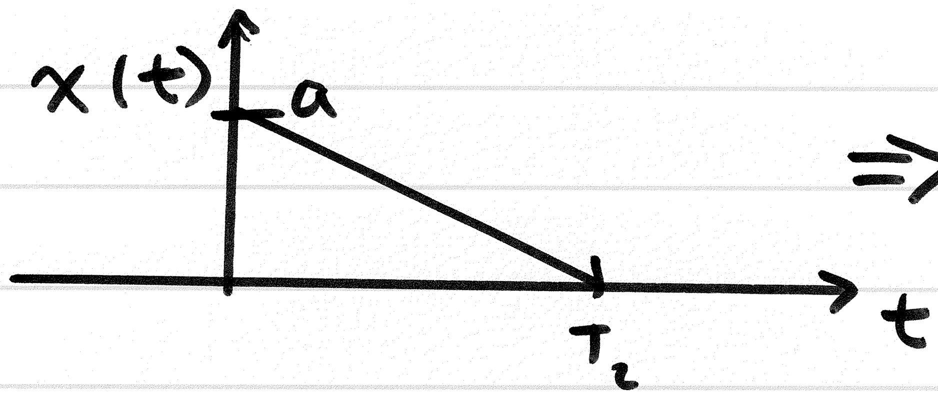
over common denominator, constant equals: $\frac{(T_1+T_2)^2}{2T_1}$

• Final answer:

$$
y(t) = \begin{cases}
0, & t < 0 \\
-\dfrac{t^2}{2T_2} + t, & 0 < t < T_1 \quad \left(\text{quadratic, partial overlap}\right) \\
-\dfrac{T_1}{T_2}t + \left(T_1 + \dfrac{T_1^2}{2T_2}\right), & T_1 < t < T_2 \quad \left(\text{linear, full overlap}\right) \\
+\dfrac{t^2}{2T_2} - \left(\dfrac{T_1+T_2}{T_2}\right)t + \left(\dfrac{(T_1+T_2)^2}{2T_2}\right), & \left(\text{quadratic, partial overlap}\right) \\
& \text{for } T_2 < t < T_1+T_2 \\
0, & \text{for } t > T_1+T_2
\end{cases}
$$



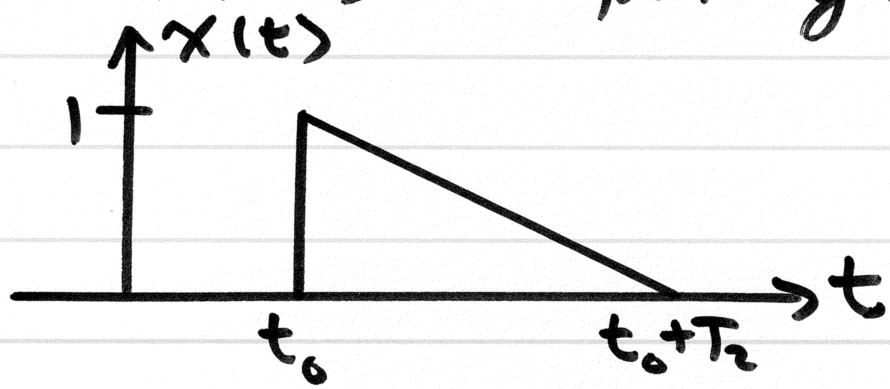
- Denote $y(t)$ on previous slide as $y_0(t)$
- What is output $y(t)$, when input is:



$\Rightarrow \quad y(t) = a\, y_0(t)$

$\Rightarrow$ linearity!

- What is output $y(t)$, when input is

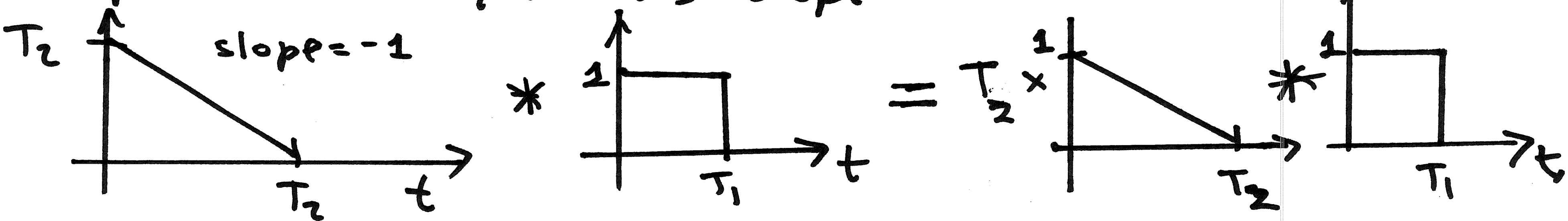


$\Rightarrow y(t) = y_0(t - t_0)$

$\Rightarrow$ TI

Convolution Result Typically Given on Front Pg of Exam 1

- ramp down triangle has slope $= -1$

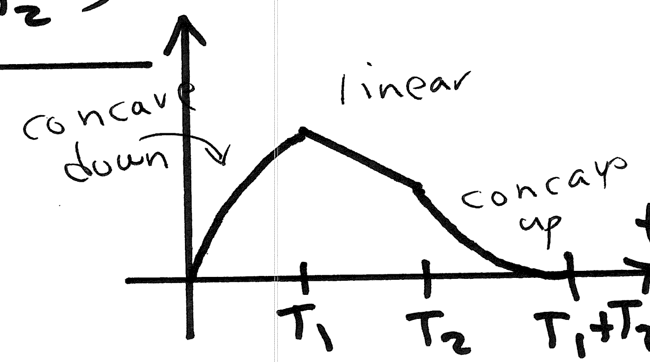


$$\underbrace{\{u(t)-u(t-T_1)\}}_{\mathrm{rect}\left(\frac{t-\frac{T_1}{2}}{T_1}\right)} * \left\{-(t-T_2)\underbrace{\left(u(t)-u(t-T_2)\right)}_{\mathrm{rect}\left(\frac{t-\frac{T_2}{2}}{T_2}\right)}\right\} =$$

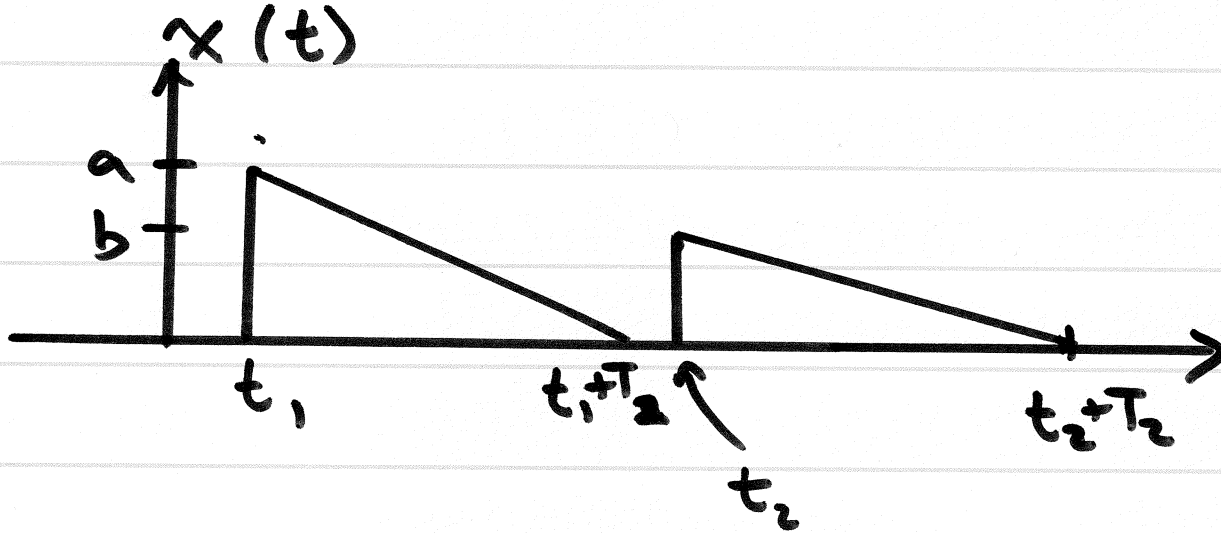
$$\left(T_2 t - t^2/2\right)\{u(t)-u(t-T_1)\}$$

$$+\left(-T_1 t + \frac{2T_1T_2+T_1^2}{2}\right)\{u(t)-u(t-T_2)\}$$

$$+\left(\frac{t^2}{2}-(T_1+T_2)t+\frac{(T_1+T_2)^2}{2}\right)\{u(t-T_2)-u(t-(T_1+T_2))\}$$



• What is output $y(t)$ when input is: 7



$$\longrightarrow \boxed{h(t) = \mathrm{rect}\left(\frac{t - T_1/2}{T_1}\right)} \longrightarrow y(t) = ?$$

Answer:

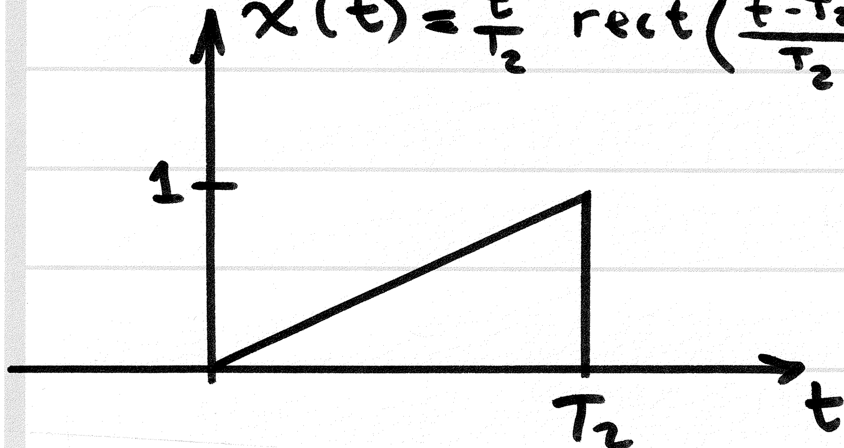
$$y(t) = a\, y_0(t - t_1) + b\, y_0(t - t_2)$$

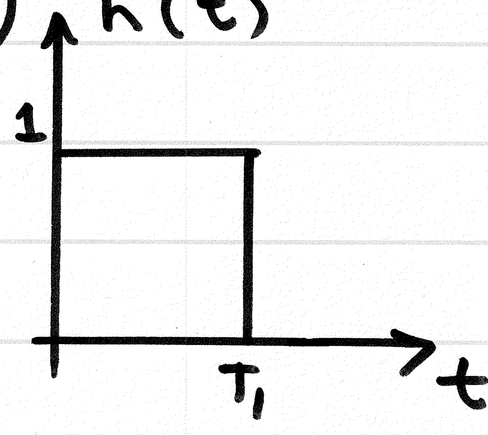
$\Rightarrow$ LTI

Another Convolution Example.

(1)

$x(t) = \frac{t}{T_2}\,\mathrm{rect}\left(\frac{t - T_2/2}{T_2}\right)$ $\quad * \quad h(t)$



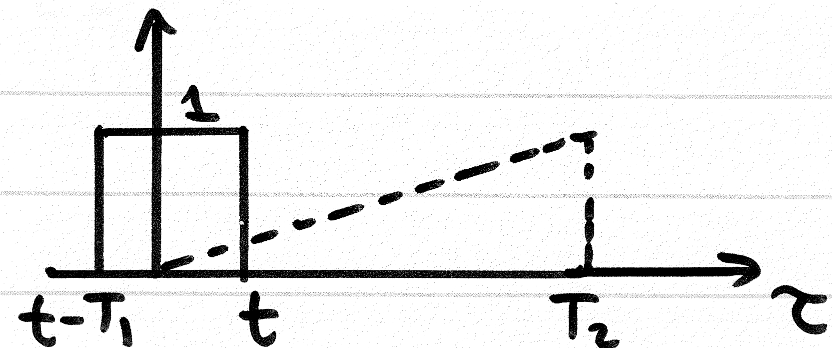


$T_2 > T_1$

$= y(t) = ?$

- for $t > 0$ and $t - T_1 < 0$

$\Rightarrow \quad 0 < t < T_1$ (partial overlap):

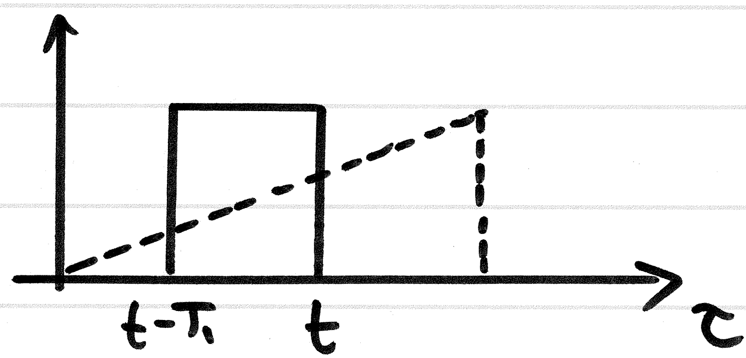


$$y(t) = \int_0^t \frac{\tau}{T_2}\, d\tau = \frac{1}{T_2} \frac{\tau^2}{2}\Bigg]_0^t = \frac{t^2}{2T_2}$$

2

• for $t - T_1 > 0$ and $t < T_2$

$\Rightarrow T_1 < t < T_2$ (full overlap)



$$y(t) = \int_{t-T_1}^{t} \frac{t}{T_2}\, d\tau = \frac{1}{2T_2}\, \tau^2 \Big]_{t-T_1}^{t}$$

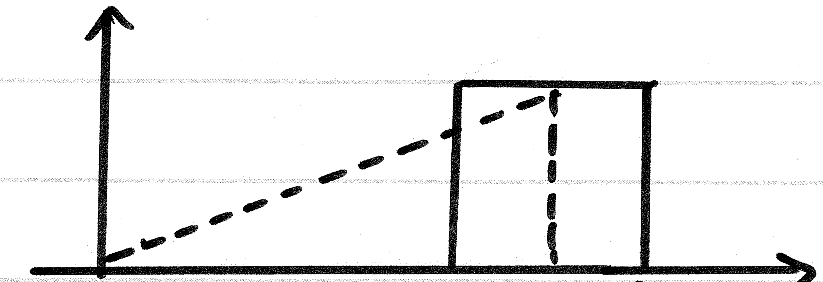
$$= \frac{1}{2T_2} \left\{ t^2 - (t - T_1)^2 \right\}$$

$$= \frac{T_1}{T_2}\, t - \frac{T_1^2}{2T_2} \quad \text{for} \quad T_1 < t < T_2$$

③

• for $t > T_2$ and $t - T_1 < T_2$

$\Rightarrow T_2 < t < T_1 + T_2$ (partial overlap):



$$y(t) = \int_{t-T_1}^{T_2} \frac{\tau}{T_2}\, d\tau = \frac{1}{2T_2} \tau^2 \Big]_{t-T_1}^{T_2}$$

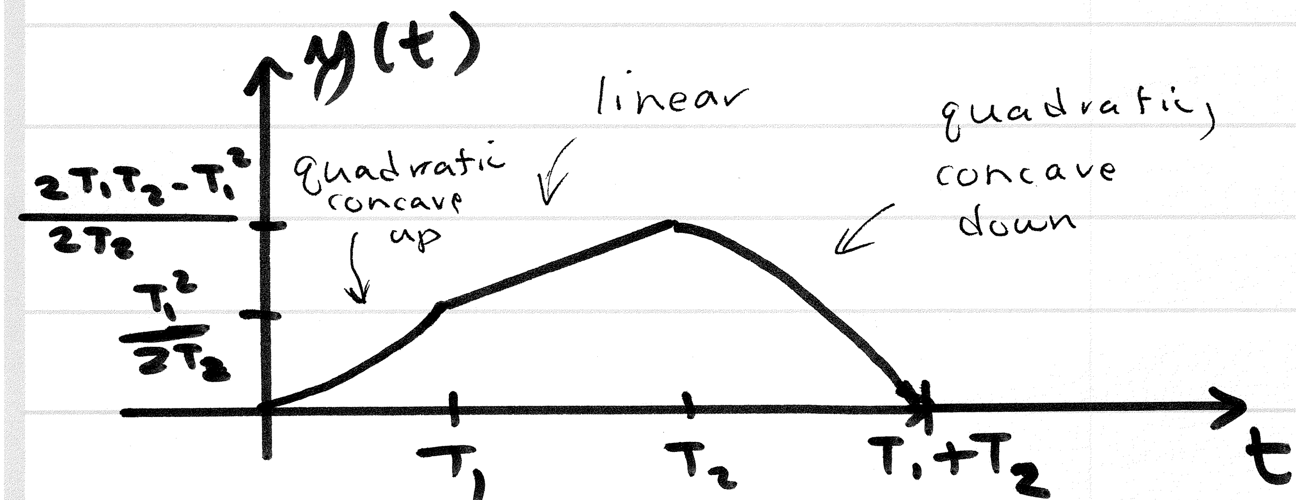
$$= \frac{1}{2T_2} \left\{ T_2^2 - (t - T_1)^2 \right\} = \frac{1}{2T_2} \left\{ T_2^2 - T_1^2 + 2T_1 t - t^2 \right\}$$

$$= -\frac{1}{2T_2} t^2 + \frac{T_1}{T_2} t + \frac{T_2^2 - T_1^2}{2T_2}$$

concave downwards

(4)

$$y(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t^2}{2T_2}, & 0 < t < T_1 \\ \dfrac{T_1}{T_2} t - \dfrac{T_1^2}{2T_2}, & T_1 < t < T_2 \\ -\dfrac{1}{2T_2} t^2 + \dfrac{T_1}{T_2} t + \dfrac{T_2^2 - T_1^2}{2T_2}, & T_2 < t < T_1 + T_2 \\ 0, & \text{for } t > T_2 \end{cases}$$

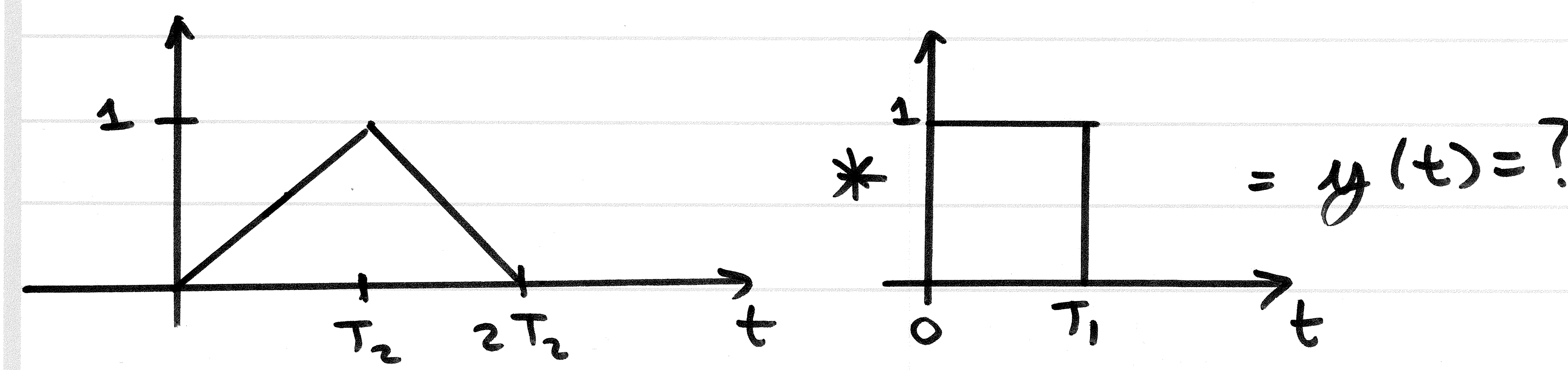


- Compare with Example 2.7 in text where $T_2 = 2T_1 = 2T$ and the triangle has a slope of 1 ⇒ so scale the $x(t)$ here by $T_2 = 2T$ ⇒ scale $y(t)$ by $2T$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{2T}{4T} t^2 = \frac{t^2}{2}, & 0 < t < T \quad ✓ \\ \frac{T(2T)}{2T} t - \frac{(2T)T^2}{4T} = Tt - \frac{T^2}{2}, & T < t < 2T \quad ✓ \\ 2T\left(-\frac{1}{4T}\right) t^2 + \frac{T}{2T}(2T) t + 2T\left(\frac{4T^2 - T^2}{4T}\right) & \quad ✓ \\ = -\frac{t^2}{2} + Tt + \frac{3}{2}T^2, & 2T < t < 3T \end{cases}$$

CHECKS! But our result is more general allowing for different widths

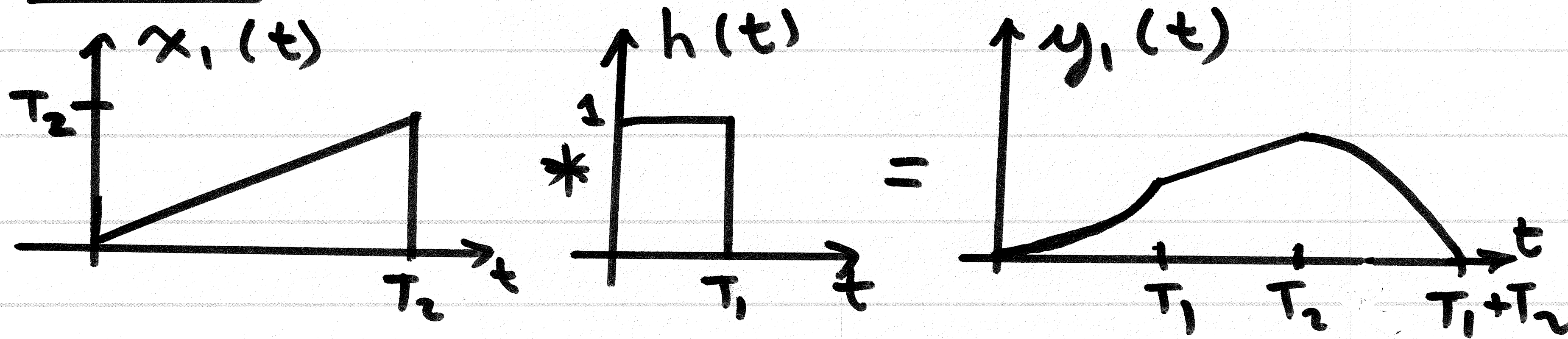
• Now you could combine both convolution examples to find the convolution of a rectangle with a triangle without much work invoking linearity and time-invariance



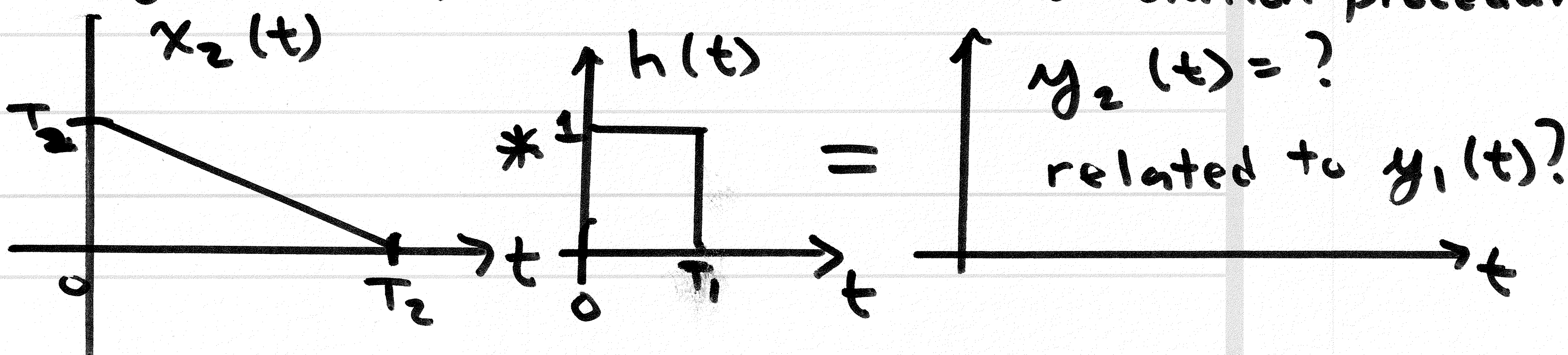
$T_2 > T_1$

Observations on Convolution of Rectangle with Ramp-Up / Ramp-Down Triangle

- Given the following convolution:



- Can we find convolution below without having to graphically work thru the convolution procedure?



- **Observations:**

$$x_2(t) = x_1(-(t-T_2))$$

$$h(t) = h(-(t-T_1))$$

- Since $y_1(t) = x_1(t) * h(t)$, it follows

$$y_1(-t) = x_1(-t) * h(-t)$$

compare

- **Now:** $y_2(t) = x_2(t) * h(t)$

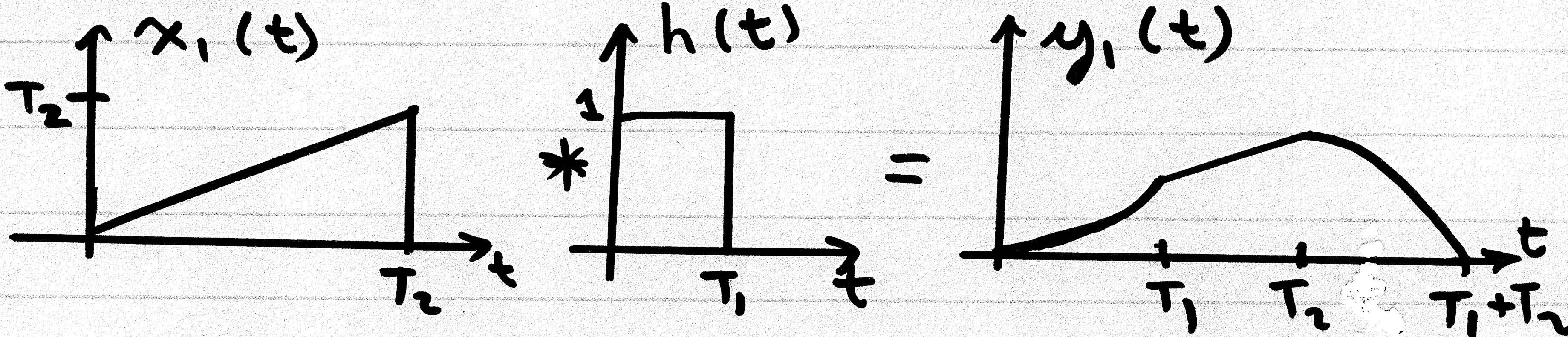
$$y_2(t) = x_1(-(t-T_2)) * h(-(t-T_1))$$

- **Thus:** $y_2(t) = y_1(-(t-[T_1+T_2]))$

$y_2(t)$ is $y_1(t)$ flipped (time-reversed) and then shifted to the right to start at $t=0$

Observations on Convolution of Rectangle with Ramp-Up / Ramp-Down Triangle

- **Given** the following convolution:



Answer: $y_2(t) = y_1\left(-\left(t-(T_1+T_2)\right)\right)$