tion and interpolation arise in a variety of important practical applications of signals and systems, including communication systems, digital audio, high-definition television, and many other applications.

Chapter 7 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining two sections contain problems belonging to the basic and advanced categories, respectively.

BASIC PROBLEMS WITH ANSWERS

7.1. A real-valued signal $x(t)$ is known to be uniquely determined by its samples when the sampling frequency is $\omega_s = 10{,}000\pi$. For what values of ω is $X(j\omega)$ guaranteed to be zero?

7.2. A continuous-time signal $x(t)$ is obtained at the output of an ideal lowpass filter with cutoff frequency $\omega_c = 1{,}000\pi$. If impulse-train sampling is performed on $x(t)$, which of the following sampling periods would guarantee that $x(t)$ can be recovered from its sampled version using an appropriate lowpass filter?
(a) $T = 0.5 \times 10^{-3}$
(b) $T = 2 \times 10^{-3}$
(c) $T = 10^{-4}$

7.3. The frequency which, under the sampling theorem, must be exceeded by the sampling frequency is called the *Nyquist rate*. Determine the Nyquist rate corresponding to each of the following signals:
(a) $x(t) = 1 + \cos(2{,}000\pi t) + \sin(4{,}000\pi t)$
(b) $x(t) = \frac{\sin(4{,}000\pi t)}{\pi t}$
(c) $x(t) = \left(\frac{\sin(4{,}000\pi t)}{\pi t}\right)^2$

7.4. Let $x(t)$ be a signal with Nyquist rate ω_0. Determine the Nyquist rate for each of the following signals:
(a) $x(t) + x(t-1)$
(b) $\frac{dx(t)}{dt}$
(c) $x^2(t)$
(d) $x(t)\cos\omega_0 t$

7.5. Let $x(t)$ be a signal with Nyquist rate ω_0. Also, let

$$y(t) = x(t)p(t-1),$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ and } T < \frac{2\pi}{\omega_0}.$$

Specify the constraints on the magnitude and phase of the frequency response of a filter that gives $x(t)$ as its output when $y(t)$ is the input.

7.6. In the system shown in Figure P7.6, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to ω_1, and $x_2(t)$ is band limited to ω_2; that is,

$$X_1(j\omega) = 0, |\omega| \geq \omega_1,$$

$$X_2(j\omega) = 0, |\omega| \geq \omega_2.$$

Determine the *maximum* sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal lowpass filter.

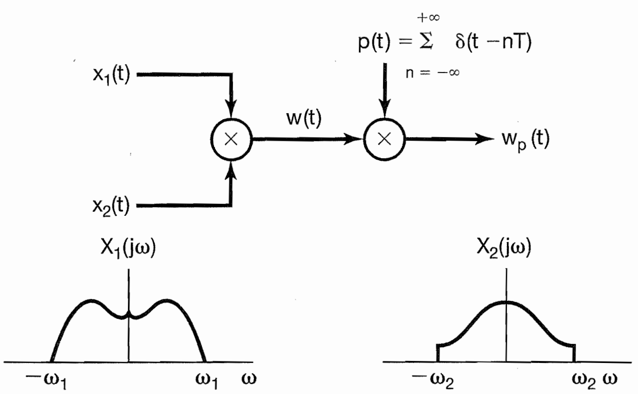

Figure P7.6

7.7. A signal $x(t)$ undergoes a zero-order hold operation with an effective sampling period T to produce a signal $x_0(t)$. Let $x_1(t)$ denote the result of a first-order hold operation on the samples of $x(t)$; i.e.,

$$x_1(t) = \sum_{n=-\infty}^{\infty} x(nT)h_1(t - nT),$$

where $h_1(t)$ is the function shown in Figure P7.7. Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input.

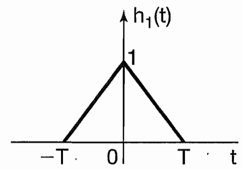

Figure P7.7

7.8. Consider a real, odd, and periodic signal $x(t)$ whose Fourier series representation may be expressed as

$$x(t) = \sum_{k=0}^{5} \left(\frac{1}{2}\right)^k \sin(k\pi t).$$

Let $\hat{x}(t)$ represent the signal obtained by performing impulse-train sampling on $x(t)$ using a sampling period of $T = 0.2$.

(a) Does aliasing occur when this impulse-train sampling is performed on $x(t)$?

(b) If $\hat{x}(t)$ is passed through an ideal lowpass filter with cutoff frequency π/T and passband gain T, determine the Fourier series representation of the output signal $g(t)$.

7.9. Consider the signal

$$x(t) = \left(\frac{\sin 50\pi t}{\pi t}\right)^2,$$

which we wish to sample with a sampling frequency of $\omega_s = 150\pi$ to obtain a signal $g(t)$ with Fourier transform $G(j\omega)$. Determine the maximum value of ω_0 for which it is guaranteed that

$$G(j\omega) = 75X(j\omega) \text{ for } |\omega| \leq \omega_0,$$

where $X(j\omega)$ is the Fourier transform of $x(t)$.

7.10. Determine whether each of the following statements is true or false:

(a) The signal $x(t) = u(t + T_0) - u(t - T_0)$ can undergo impulse-train sampling without aliasing, provided that the sampling period $T < 2T_0$.

(b) The signal $x(t)$ with Fourier transform $X(j\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ can undergo impulse-train sampling without aliasing, provided that the sampling period $T < \pi/\omega_0$.

(c) The signal $x(t)$ with Fourier transform $X(j\omega) = u(\omega) - u(\omega - \omega_0)$ can undergo impulse-train sampling without aliasing, provided that the sampling period $T < 2\pi/\omega_0$.

7.11. Let $x_c(t)$ be a continuous-time signal whose Fourier transform has the property that $X_c(j\omega) = 0$ for $|\omega| \geq 2{,}000\pi$. A discrete-time signal

$$x_d[n] = x_c(n(0.5 \times 10^{-3}))$$

is obtained. For each of the following constraints on the Fourier transform $X_d(e^{j\omega})$ of $x_d[n]$, determine the corresponding constraint on $X_c(j\omega)$:

(a) $X_d(e^{j\omega})$ is real.

(b) The maximum value of $X_d(e^{j\omega})$ over all ω is 1.

(c) $X_d(e^{j\omega}) = 0$ for $\frac{3\pi}{4} \le |\omega| \le \pi$.

(d) $X_d(e^{j\omega}) = X_d(e^{j(\omega-\pi)})$.

7.12. A discrete-time signal $x_d[n]$ has a Fourier transform $X_d(e^{j\omega})$ with the property that $X_d(e^{j\omega}) = 0$ for $3\pi/4 \le |\omega| \le \pi$. The signal is converted into a continuous-time signal

$$x_c(t) = T \sum_{n=-\infty}^{\infty} x_d[n] \frac{\sin \frac{\pi}{T}(t - nT)}{\pi(t - nT)},$$

where $T = 10^{-3}$. Determine the values of ω for which the Fourier transform $X_c(j\omega)$ of $x_c(t)$ is guaranteed to be zero.

7.13. With reference to the filtering approach illustrated in Figure 7.24, assume that the sampling period used is T and the input $x_c(t)$ is band limited, so that $X_c(j\omega) = 0$ for $|\omega| \ge \pi/T$. If the overall system has the property that $y_c(t) = x_c(t-2T)$, determine the impulse response $h[n]$ of the discrete-time filter in Figure 7.24.

7.14. Repeat the previous problem, except this time assume that

$$y_c(t) = \frac{d}{dt} x_c\left(t - \frac{T}{2}\right).$$

7.15. Impulse-train sampling of $x[n]$ is used to obtain

$$g[n] = \sum_{k=-\infty}^{\infty} x[n]\delta[n - kN].$$

If $X(e^{j\omega}) = 0$ for $3\pi/7 \le |\omega| \le \pi$, determine the largest value for the sampling interval N which ensures that no aliasing takes place while sampling $x[n]$.

7.16. The following facts are given about the signal $x[n]$ and its Fourier transform:

1. $x[n]$ is real.
2. $X(e^{j\omega}) \ne 0$ for $0 < \omega < \pi$.
3. $x[n]\sum_{k=-\infty}^{\infty}\delta[n - 2k] = \delta[n]$.

Determine $x[n]$. You may find it useful to note that the signal $(\sin \frac{\pi}{2} n)/(\pi n)$ satisfies two of these conditions.

7.17. Consider an ideal discrete-time bandstop filter with impulse response $h[n]$ for which the frequency response in the interval $-\pi \le \omega \le \pi$ is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{4} \text{ and } |\omega| \ge \frac{3\pi}{4} \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the frequency response of the filter whose impulse response is $h[2n]$.

7.18. Suppose the impulse response of an ideal discrete-time lowpass filter with cutoff frequency $\pi/2$ is interpolated (in accordance with Figure 7.37) to obtain an upsampling by a factor of 2. What is the frequency response corresponding to this upsampled impulse response?

7.19. Consider the system shown in Figure P7.19, with input $x[n]$ and the corresponding output $y[n]$. The zero-insertion system inserts two points with zero amplitude between each of the sequence values in $x[n]$. The decimation is defined by

$$y[n] = w[5n],$$

where $w[n]$ is the input sequence for the decimation system. If the input is of the form

$$x[n] = \frac{\sin \omega_1 n}{\pi n},$$

determine the output $y[n]$ for the following values of ω_1:

(a) $\omega_1 \le \frac{3\pi}{5}$

(b) $\omega_1 > \frac{3\pi}{5}$

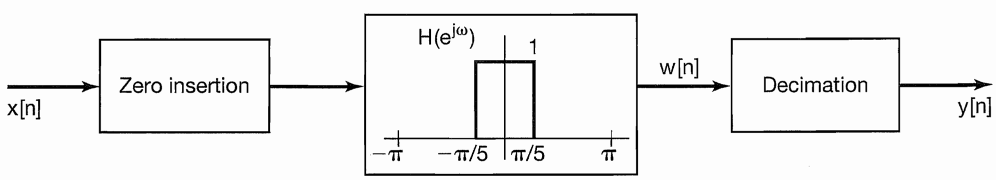

Figure P7.19

7.20. Two discrete-time systems S_1 and S_2 are proposed for implementing an ideal lowpass filter with cutoff frequency $\pi/4$. System S_1 is depicted in Figure P7.20(a). System S_2 is depicted in Figure P7.20(b). In these figures, S_A corresponds to a zero-insertion system that inserts one zero after every input sample, while S_B corresponds to a decimation system that extracts every second sample of its input.

(a) Does the proposed system S_1 correspond to the desired ideal lowpass filter?

(b) Does the proposed system S_2 correspond to the desired ideal lowpass filter?

$$y_p(t) = \sum_{n=-\infty}^{+\infty} y(nT)\delta(t - nT).$$

Specify the range of values for the sampling period T which ensures that $y(t)$ is recoverable from $y_p(t)$.

7.23. Shown in Figure P7.23 is a system in which the sampling signal is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in the figure.

(a) For $\Delta < \pi/(2\omega_M)$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

(b) For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $x_p(t)$.

(c) For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $y(t)$.

(d) What is the *maximum* value of Δ in relation to ω_M for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$?

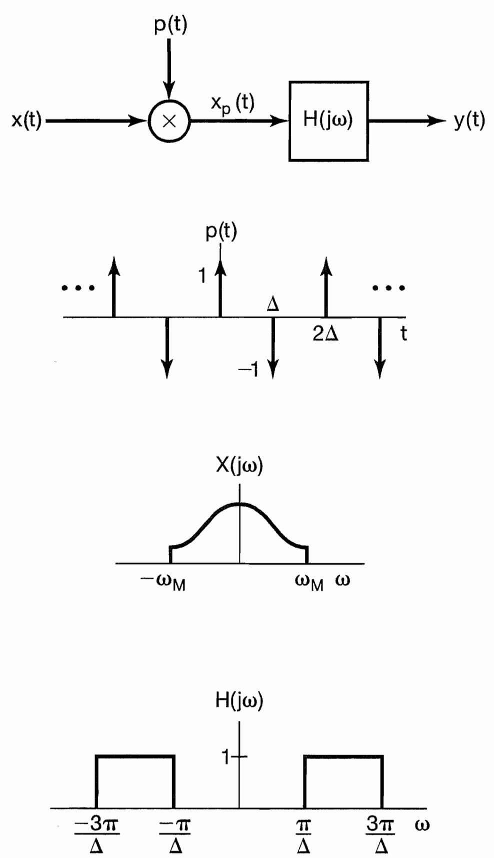

Figure P7.23

7.24. Shown in Figure P7.24 is a system in which the input signal is multiplied by a periodic square wave. The period of $s(t)$ is T. The input signal is band limited with $|X(j\omega)| = 0$ for $|\omega| \geq \omega_M$.

By considering the values of α for which $[\sin(\alpha)]/\alpha = 0$, show from eq. (P7.25–1) that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of k.

7.26. The sampling theorem, as we have derived it, states that a signal $x(t)$ must be sampled at a rate greater than its bandwidth (or equivalently, a rate greater than twice its highest frequency). This implies that if $x(t)$ has a spectrum as indicated in Figure P7.26(a) then $x(t)$ must be sampled at a rate greater than $2\omega_2$. However, since the signal has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than twice the highest frequency could be used. A signal whose energy is concentrated in a frequency band is often referred to as a *bandpass signal*. There are a variety of techniques for sampling such signals, generally referred to as *bandpass-sampling* techniques.

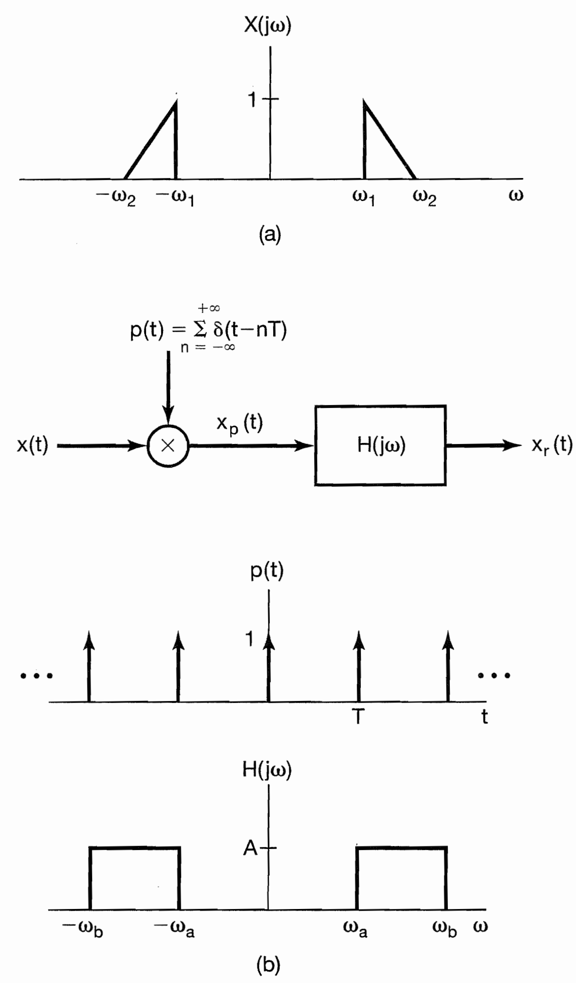

Figure P7.26

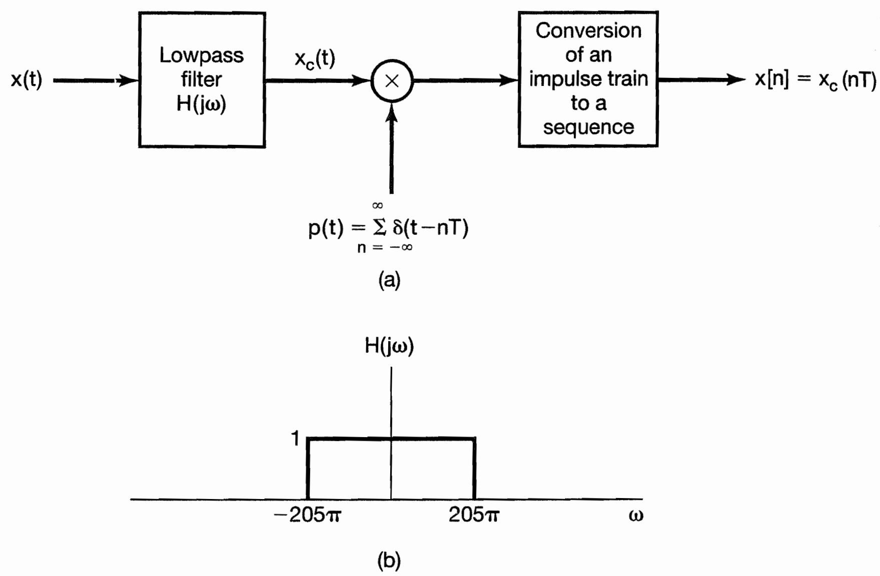

Figure P7.28

7.29. Figure P7.29(a) shows the overall system for filtering a continuous-time signal using a discrete-time filter. If $X_c(j\omega)$ and $H(e^{j\omega})$ are as shown in Figure P7.29(b), with $1/T = 20$ kHz, sketch $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$.

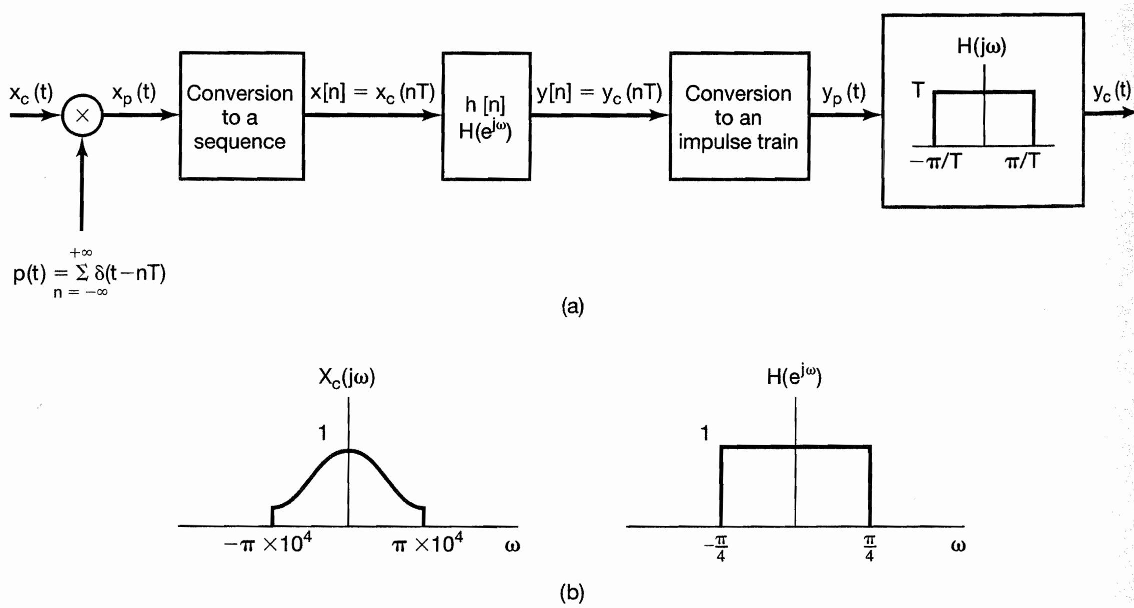

Figure P7.29

$$g[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n-1-4k]$$

is generated. Specify the frequency response $H(e^{j\omega})$ of a lowpass filter that produces $x[n]$ as output when $g[n]$ is the input.

7.33. A signal $x[n]$ with Fourier transform $X(e^{j\omega})$ has the property that

$$\left(x[n] \sum_{k=-\infty}^{\infty} \delta[n-3k]\right) * \left(\frac{\sin \frac{\pi}{3} n}{\frac{\pi}{3} n}\right) = x[n].$$

For what values of ω is it guaranteed that $X(e^{j\omega}) = 0$?

7.34. A real-valued discrete-time signal $x[n]$ has a Fourier transform $X(e^{j\omega})$ that is zero for $3\pi/14 \leq |\omega| \leq \pi$. The nonzero portion of the Fourier transform of one period of $X(e^{j\omega})$ can be made to occupy the region $|\omega| < \pi$ by first performing upsampling by a factor of L and then performing downsampling by a factor of M. Specify the values of L and M.

7.35. Consider a discrete-time sequence $x[n]$ from which we form two new sequences, $x_p[n]$ and $x_d[n]$, where $x_p[n]$ corresponds to sampling $x[n]$ with a sampling period of 2 and $x_d[n]$ corresponds to decimating $x[n]$ by a factor of 2, so that

$$x_p[n] = \begin{cases} x[n], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 1, \pm 3, \ldots \end{cases}$$

and

$$x_d[n] = x[2n].$$

(a) If $x[n]$ is as illustrated in Figure P7.35(a), sketch the sequences $x_p[n]$ and $x_d[n]$.
(b) If $X(e^{j\omega})$ is as shown in Figure P7.35(b), sketch $X_p(e^{j\omega})$ and $X_d(e^{j\omega})$.

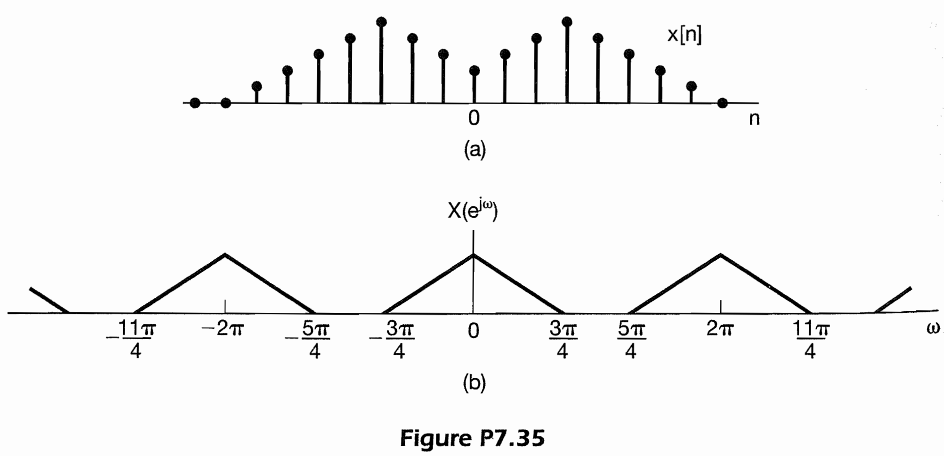

Figure P7.35

ADVANCED PROBLEMS

7.36 Let $x(t)$ be a band-limited signal such that $X(j\omega) = 0$ for $|\omega| \geq \frac{\pi}{T}$.
(a) If $x(t)$ is sampled using a sampling period T, determine an interpolating function

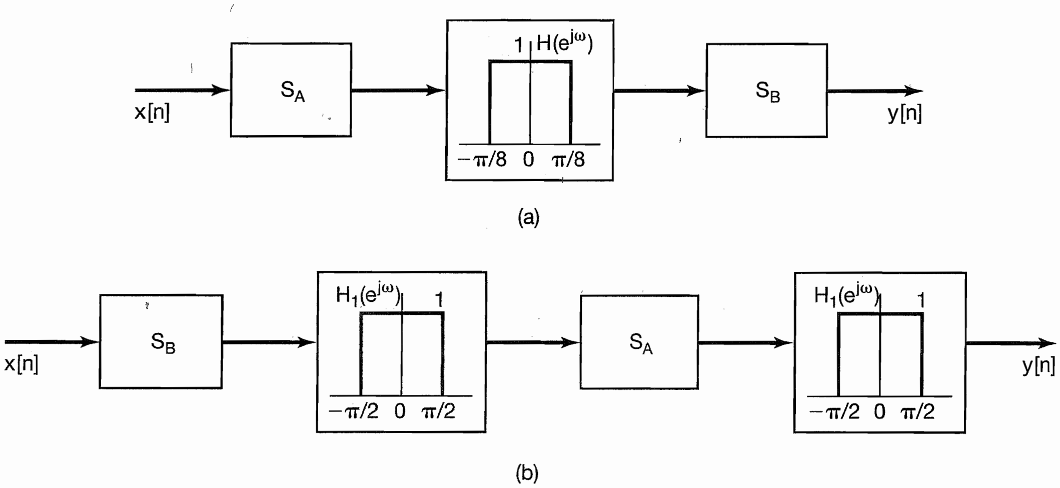

Figure P7.20

BASIC PROBLEMS

7.21. A signal $x(t)$ with Fourier transform $X(j\omega)$ undergoes impulse-train sampling to generate

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

where $T = 10^{-4}$. For each of the following sets of constraints on $x(t)$ and/or $X(j\omega)$, does the sampling theorem (see Section 7.1.1) guarantee that $x(t)$ can be recovered exactly from $x_p(t)$?

(a) $X(j\omega) = 0$ for $|\omega| > 5000\pi$
(b) $X(j\omega) = 0$ for $|\omega| > 15000\pi$
(c) $\mathcal{R}e\{X(j\omega)\} = 0$ for $|\omega| > 5000\pi$
(d) $x(t)$ real and $X(j\omega) = 0$ for $\omega > 5000\pi$
(e) $x(t)$ real and $X(j\omega) = 0$ for $\omega < -15000\pi$
(f) $X(j\omega) * X(j\omega) = 0$ for $|\omega| > 15000\pi$
(g) $|X(j\omega)| = 0$ for $\omega > 5000\pi$

7.22. The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$, that is,

$$y(t) = x_1(t) * x_2(t)$$

where

$$X_1(j\omega) = 0 \quad \text{for } |\omega| > 1000\pi$$
$$X_2(j\omega) = 0 \quad \text{for } |\omega| > 2000\pi.$$

Impulse-train sampling is performed on $y(t)$ to obtain

(a) For $\Delta = T/3$, determine, in terms of ω_M, the maximum value of T for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

(b) For $\Delta = T/4$, determine, in terms of ω_M, the maximum value of T for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

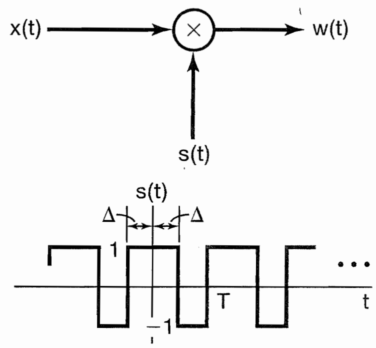

Figure P7.24

7.25. In Figure P7.25 is a sampler, followed by an ideal lowpass filter, for reconstruction of $x(t)$ from its samples $x_p(t)$. From the sampling theorem, we know that if $\omega_s = 2\pi/T$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then the reconstructed signal $x_r(t)$ will exactly equal $x(t)$. If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will *not* equal $x(t)$. We seek to show in this problem that if $\omega_c = \omega_s/2$, then for any choice of T, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants; that is,

$$x_r(kT) = x(kT),\ k = 0, \pm 1, \pm 2, \ldots.$$

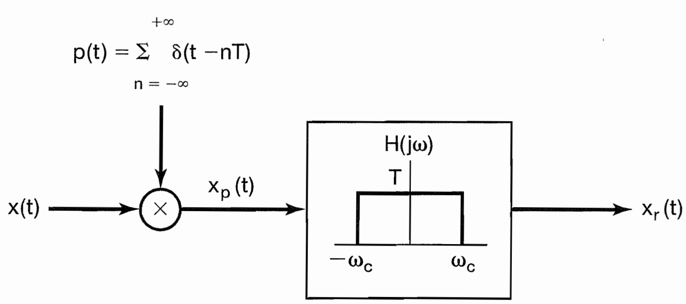

Figure P7.25

To obtain this result, consider eq. (7.11), which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)T\frac{\omega_c}{\pi}\frac{\sin[\omega_c(t-nT)]}{\omega_c(t-nT)}.$$

With $\omega_c = \omega_s/2$, this becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)}. \qquad \text{(P7.25–1)}$$

To examine the possibility of sampling a bandpass signal as a rate less than the total bandwidth, consider the system shown in Figure P7.26(b). Assuming that $\omega_1 > \omega_2 - \omega_1$, find the maximum value of T and the values of the constants A, ω_a, and ω_b such that $x_r(t) = x(t)$.

7.27. In Problem 7.26, we considered one procedure for bandpass sampling and reconstruction. Another procedure, used when $x(t)$ is real, consists of multiplying $x(t)$ by a complex-exponential and then sampling the product. The sampling system is shown in Figure P7.27(a). With $x(t)$ real and with $X(j\omega)$ nonzero only for $\omega_1 < |\omega| < \omega_2$, the frequency is chosen to be $\omega_0 = (1/2)(\omega_1 + \omega_2)$, and the lowpass filter $H_1(j\omega)$ has cutoff frequency $(1/2)(\omega_2 - \omega_1)$.

(a) For $X(j\omega)$ as shown in Figure P7.27(b), sketch $X_p(j\omega)$.

(b) Determine the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$.

(c) Determine a system to recover $x(t)$ from $x_p(t)$.

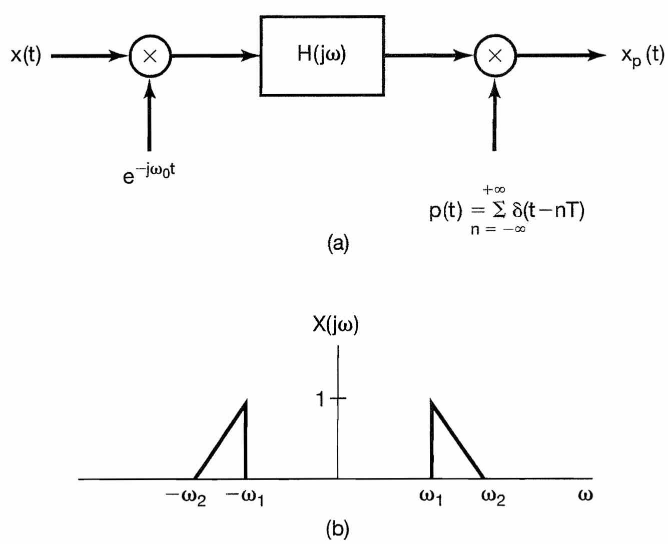

Figure P7.27

7.28. Figure P7.28(a) shows a system that converts a continuous-time signal to a discrete-time signal. The input $x(t)$ is periodic with a period of 0.1 second. The Fourier series coefficients of $x(t)$ are

$$a_k = \left(\frac{1}{2}\right)^{|k|}, \quad -\infty < k < +\infty.$$

The lowpass filter $H(j\omega)$ has the frequency response shown in Figure P7.28(b). The sampling period $T = 5 \times 10^{-3}$ second.

(a) Show that $x[n]$ is a periodic sequence, and determine its period.

(b) Determine the Fourier series coefficients of $x[n]$.

7.30. Figure P7.30 shows a system consisting of a continuous-time LTI system followed by a sampler, conversion to a sequence, and an LTI discrete-time system. The continuous-time LTI system is causal and satisfies the linear, constant-coefficient differential equation

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t).$$

The input $x_c(t)$ is a unit impulse $\delta(t)$.

(a) Determine $y_c(t)$.

(b) Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ such that $w[n] = \delta[n]$.

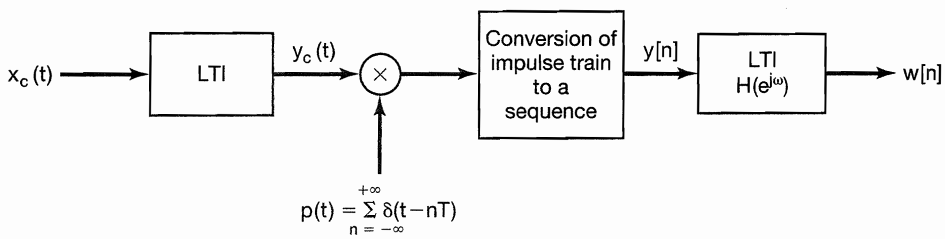

Figure P7.30

7.31. Shown in Figure P7.31 is a system that processes continuous-time signals using a digital filter $h[n]$ that is linear and causal with difference equation

$$y[n] = \frac{1}{2}y[n-1] + x[n].$$

For input signals that are band limited such that $X_c(j\omega) = 0$ for $|\omega| > \pi/T$, the system in the figure is equivalent to a continuous-time LTI system.

Determine the frequency response $H_c(j\omega)$ of the equivalent overall system with input $x_c(t)$ and output $y_c(t)$.

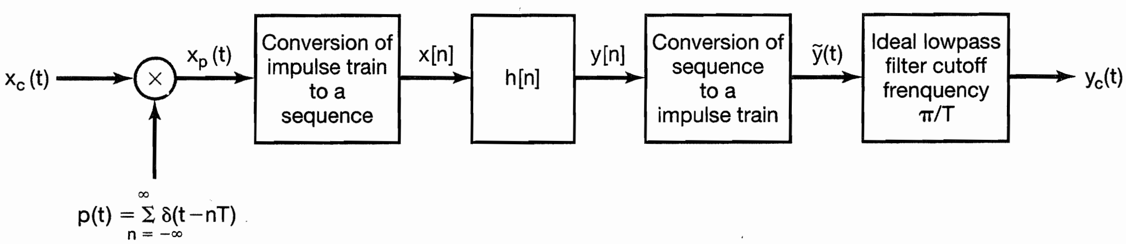

Figure P7.31

7.32. A signal $x[n]$ has a Fourier transform $X(e^{j\omega})$ that is zero for $(\pi/4) \leq |\omega| \leq \pi$. Another signal

$g(t)$ such that

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT).$$

(b) Is the function $g(t)$ unique?

7.37. A signal limited in bandwidth to $|\omega| < W$ can be recovered from nonuniformly spaced samples as long as the average sample density is $2(W/2\pi)$ samples per second. This problem illustrates a particular example of nonuniform sampling. Assume that in Figure P7.37(a):

1. $x(t)$ is band limited; $X(j\omega) = 0$, $|\omega| > W$.
2. $p(t)$ is a nonuniformly spaced periodic pulse train, as shown in Figure P7.37(b).
3. $f(t)$ is a periodic waveform with period $T = 2\pi/W$. Since $f(t)$ multiplies an impulse train, only its values $f(0) = a$ and $f(\Delta) = b$ at $t = 0$ and $t = \Delta$, respectively, are significant.
4. $H_1(j\omega)$ is a 90° phase shifter; that is,

$$H_1(j\omega) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}.$$

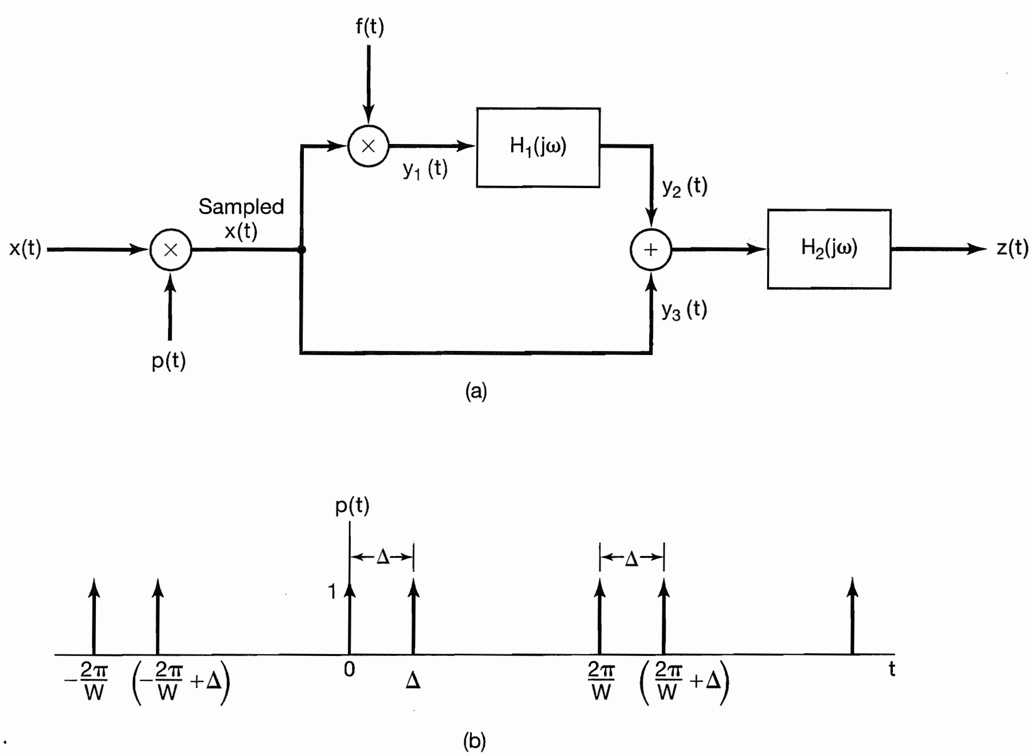

Figure P7.37