Prob. 5.35 All-Pass Filters ⟹ "Pet Problem"

$$y[n] - a\,y[n-1] = b\,x[n] + x[n-1]$$

Take DTFT of both sides:

$$Y(\omega)\left(1 - a e^{-j\omega}\right) = X(\omega)\left(b + e^{-j\omega}\right)$$

Convolution prop. dictates $Y(\omega) = H(\omega)\,X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}}$$

So, we have DTFT of $h[n] = H(\omega)$
frequency response of system
without even determining
impulse response $h[n]$

Consider $b = -a$: (where a is real-valued)

(2)

$$H(\omega) = \frac{-a + e^{-j\omega}}{1 - a e^{-j\omega}} = e^{-j\omega}\frac{(1 - a e^{j\omega})}{1 - a e^{-j\omega}}$$

since a is real-valued

$$\frac{1 - a e^{j\omega}}{1 - a e^{-j\omega}} = \frac{c}{c^*}$$

for any complex no. c, $\frac{c}{c^*}$ has magnitude 1

$\Rightarrow$ easy to see in polar form

$$\frac{c}{c^*} = \frac{|c| e^{j\angle c}}{|c| e^{-j\angle c}} = e^{j2\angle c}$$

and since $|ab| = |a||b|$, we have

$$|H(\omega)| = |e^{-j\omega}| \left|\frac{1 - a e^{j\omega}}{1 - a e^{-j\omega}}\right| = 1 \text{ for all } \omega$$

• **Thus:** $y[n] - a\,y[n-1] = -a\,x[n] + x[n-1]$

is an all-pass (magnitude) filter for any value of a (real-valued) $|H(\omega)| = 1 \;\; \forall \omega$

• Thus, for any sinewave into this system, amplitude is unchanged $\Rightarrow$ just phase changes

$$A e^{j(\omega_0 n + \theta)} \longrightarrow \boxed{\text{All-Pass Filter}} \longrightarrow A e^{j(\omega_0 n + \theta + \phi(\omega_0))}$$

$$\phi(\omega_0) = \angle H(\omega_0)$$

• For arbitrary input $x[n]$, consider:

$$x[n] \longrightarrow \boxed{\text{All-Pass Filter}} \longrightarrow y[n]$$

$\boxed{\text{Consider Parseval's Theorem}}$

<u>note</u>: all-pass filter does <u>NOT</u> mean $y[n] = x[n]$

$$x[n] \longrightarrow \boxed{\text{All-Pass Filter}} \longrightarrow y[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad\qquad E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \qquad\qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(\omega)|^2 d\omega$$

- since $Y(\omega) = H(\omega)X(\omega)$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

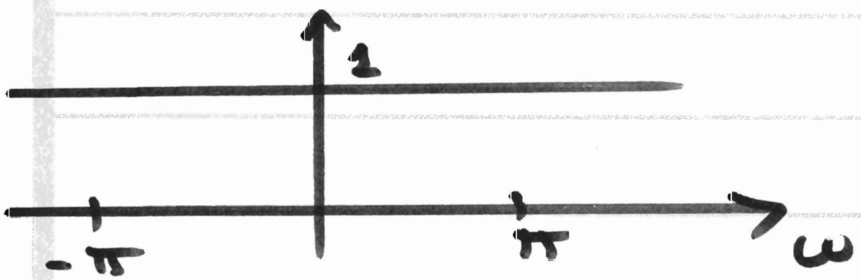
$|Y(\omega)| = |X(\omega)|$ For all-pass filter

Thus, $E_y = E_x$ for all-pass filter

(pet problem :-))

$$H(\omega) = \frac{-a + e^{-j\omega}}{1 - a e^{-j\omega}} = e^{-j\omega}\,\frac{(1 - a e^{j\omega})}{1 - a e^{-j\omega}}$$ 5

$|H(\omega)| = 1 \quad \forall \omega$



$\angle H(\omega) = ?$

$= -\omega + 2\angle(1 - a e^{j\omega})$

$1 - a e^{j\omega}$

$= (1 - a\cos\omega) - j a \sin\omega$

$$\angle(1 - a e^{j\omega}) = -\tan^{-1}\left\{\frac{a\sin\omega}{1 - a\cos\omega}\right\}$$

Thus:

$$\angle H(\omega) = -\omega - 2\tan^{-1}\left\{\frac{a\sin\omega}{1 - a\cos\omega}\right\}$$

$-1 < a < 1$ for stability (later: Chap. 10)

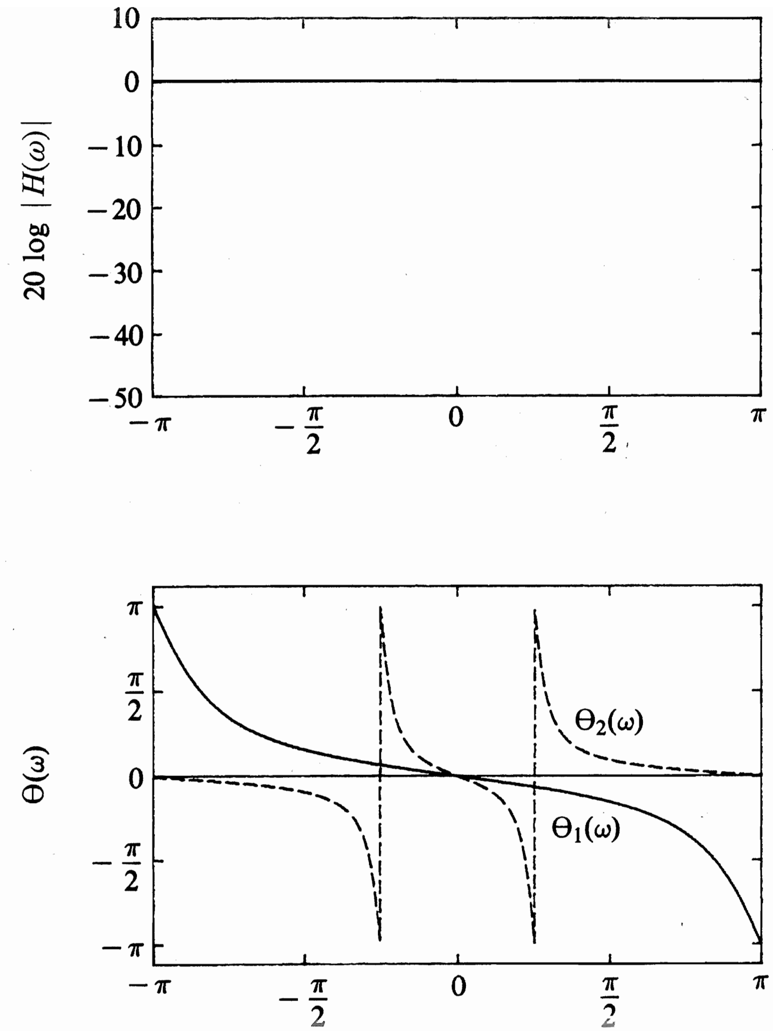


Figure 5.4.17
Frequency response characteristics of an all-pass filter with system functions (1) $H(z) = (0.6 + z^{-1})/(1 + 0.6z^{-1})$, (2) $H(z) = (r^2 - 2r\cos\omega_0 z^{-1} + z^{-2})/(1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})$, $r = 0.9$, $\omega_0 = \pi/4$.