Chap 5 DTFT Problems Help/Examples

Prob. 5.21 (a) Recall basic DTFT pair and basic DTFT property (Tables 5.1 and 5.2)

$$u[n]-u[n-N] \xleftrightarrow{DTFT} e^{-j\frac{(N-1)}{2}\omega}\,\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$x[n-n_0] \xleftrightarrow{DTFT} X(\omega)e^{-j\omega n_0}$$

$$x[n]=u[n-2]-u[n-6]=\tilde{x}[n-2]$$

where: $\tilde{x}[n]=u[n]-u[n-4]$

Thus:

$$u[n-2]-u[n-6] \xleftrightarrow{DTFT} e^{-j2\omega}\,e^{-j\frac{(4-1)}{2}\omega}\,\frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$=e^{-j\frac{7}{2}\omega}\,\frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}$$

5.21 (b) $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1] = \left(\frac{1}{2}\right)^{-n} u[-(n+1)]$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-(n+1)} u[-(n+1)]$$

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

$$x[-n] \xleftrightarrow{DTFT} X(-\omega) \qquad x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(\omega)$$

$n_0 = -1$ here

End result:

$$X(\omega) = \frac{1}{2} e^{-j\omega(-1)} \frac{1}{1-\frac{1}{2}e^{-j(-\omega)}}$$

$$= \frac{1}{2} \frac{e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}$$

5.21 (c) $x[n] = \left(\frac{1}{3}\right)^{|n|} u[-n-2] = \left(\frac{1}{3}\right)^{|n|} \underbrace{u[-(n+2)]}$

$x[n] = \left(\frac{1}{3}\right)^{-n} u[-(n+2)]$ $\Longrightarrow$ turns on at $n=-\infty$

shuts off at $n=-1$

$= \left(\frac{1}{3}\right)^{2}\left(\frac{1}{3}\right)^{-(n+2)} u[-(n+2)]$

since $|n| = -n$ for $n<0$

From previous problem:

$$X(\omega) = \frac{1}{9} e^{-j\omega(-2)} \frac{1}{1-\frac{1}{3}e^{-j(-\omega)}}$$

$$= \frac{1}{9} \frac{e^{j2\omega}}{1-\frac{1}{3}e^{j\omega}}$$

Prob. 5.21 (d) $x[n] = 2^n \sin(\frac{\pi}{4}n) u[-n]$

$$= 2^n u[-n] \left\{ \frac{1}{2j} e^{j\frac{\pi}{4}n} - \frac{1}{2j} e^{-j\frac{\pi}{4}n} \right\}$$

$$= \left(\frac{1}{2}\right)^{-n} u[-n] \left\{ \text{"} \qquad \text{"} \right\}$$

From 5.21 (b) and (c) plus freq. shift prop. and linearity $e^{j\omega_0 n} x[n] \overset{DTFT}{\longleftrightarrow} X(\omega - \omega_0)$

$$X(\omega) = \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{-j(-(\omega - \frac{\pi}{4}))}} - \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{-j(-(\omega + \frac{\pi}{4}))}}$$

$$= \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{j(\omega - \pi/4)}} - \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{j(\omega + \pi/4)}}$$

5.21 (e) $x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$

Example 5.2 $\quad a^{|n|} \xleftrightarrow{\text{DTFT}} \dfrac{1-a^2}{1-2a\cos\omega + a^2}$

pp 364-365

$$x[n] = \left(\frac{1}{2}\right)^{|n|} \frac{1}{2} e^{j\frac{\pi}{8}n} e^{-j\frac{\pi}{8}} + \frac{1}{2}\left(\frac{1}{2}\right)^{|n|} e^{j\frac{\pi}{8}} e^{-j\frac{\pi}{8}n}$$

Thus:

$$X(\omega) = \frac{1}{2} e^{-j\frac{\pi}{8}} \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\left(\omega-\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)^2}$$

$$+ \frac{1}{2} e^{j\frac{\pi}{8}} \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\left(\omega+\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)^2}$$

$$= \frac{3}{8}\left\{ \frac{e^{-j\pi/8}}{\frac{5}{4} - \cos\left(\omega - \frac{\pi}{8}\right)} + \frac{e^{j\pi/8}}{\frac{5}{4} - \cos\left(\omega + \frac{\pi}{8}\right)} \right\}$$

Prob. 5.21 (f) $x[n] = \begin{cases} n, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$

$x[n] = n\,\tilde{x}[n]$ where $\tilde{x}[n] = u[n+3] - u[n-4]$

The DTFT of $\tilde{x}[n]$ is $\tilde{X}(\omega) = \dfrac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$

Table 5.2

and using this prop. from Table 5.1

$$n\,x[n] \xleftrightarrow{DTFT} j\,\frac{dX(\omega)}{d\omega}$$

End result:

$$X(\omega) = j\,\frac{d}{d\omega}\left\{\frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right\}$$

$$= j\,\frac{1}{\sin^2\left(\frac{\omega}{2}\right)}\left\{\frac{7}{2}\cos\left(\frac{7}{2}\omega\right)\sin\left(\frac{1}{2}\omega\right) - \sin\left(\frac{7}{2}\omega\right)\frac{1}{2}\cos\left(\frac{\omega}{2}\right)\right\}$$

5.21 (g) $x[n] = \sin\left(\frac{\pi}{2}n\right) + \cos(n)$

$$X(\omega) = \frac{2\pi}{2j}\,\delta\left(\omega - \frac{\pi}{2}\right) - \frac{2\pi}{2j}\,\delta\left(\omega + \frac{\pi}{2}\right) + \frac{2\pi}{2}\,\delta(\omega - 1) + \frac{2\pi}{2}\,\delta(\omega + 1)$$

for $-\pi < \omega < \pi$

repeats every 2π

5.21 (h) $x[n] = \sin\left(\frac{5\pi}{3}n\right) + \cos\left(\frac{7\pi}{3}n\right)$

$$= \sin\left(\left(\frac{5\pi}{3} - \frac{6\pi}{3}\right)n\right) + \cos\left(\left(\frac{7\pi}{3} - \frac{6\pi}{3}\right)n\right)$$

$$= -\sin\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}n\right)$$

$$X(\omega) = 2\pi\left(\frac{1}{2} - \frac{1}{2j}\right)\delta\left(\omega - \frac{\pi}{3}\right) + \left(\frac{1}{2} + \frac{1}{2j}\right)\delta\left(\omega + \frac{\pi}{3}\right)2\pi$$

for $-\pi < \omega < \pi$ repeats every 2π

5.21 (i) $x[n] = \tilde{x}[n-6]$ $\tilde{x}[n] = u[n] - u[n-5]$

From 5.21 (a)

$$X(\omega) = \underbrace{e^{-j6\omega} \, e^{-j\frac{(5-1)}{2}\omega}}_{} \frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$= e^{-j8\omega}$$

5.21 (j) $x[n] = (n-1)\left(\frac{1}{3}\right)^{|n|}$

$$= n\left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{3}\right)^{|n|}$$

$$X(\omega) = j\frac{d}{d\omega}\left\{\frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2 - \frac{2}{3}\cos(\omega)}\right\} - \frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2 - \frac{2}{3}\cos\omega}$$

simplification left to the reader :)

5.21 (k)

$$x[n] = \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n}\cos\left(\frac{7\pi}{2}n\right)$$

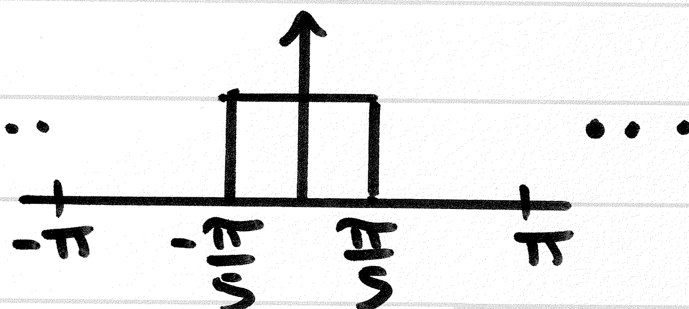
$$\cos\left(\frac{7\pi}{2}n\right) = \cos\left(\left(\frac{7\pi}{2} - \frac{8\pi}{2}\right)n\right) = \cos\left(-\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right)$$

Thus:

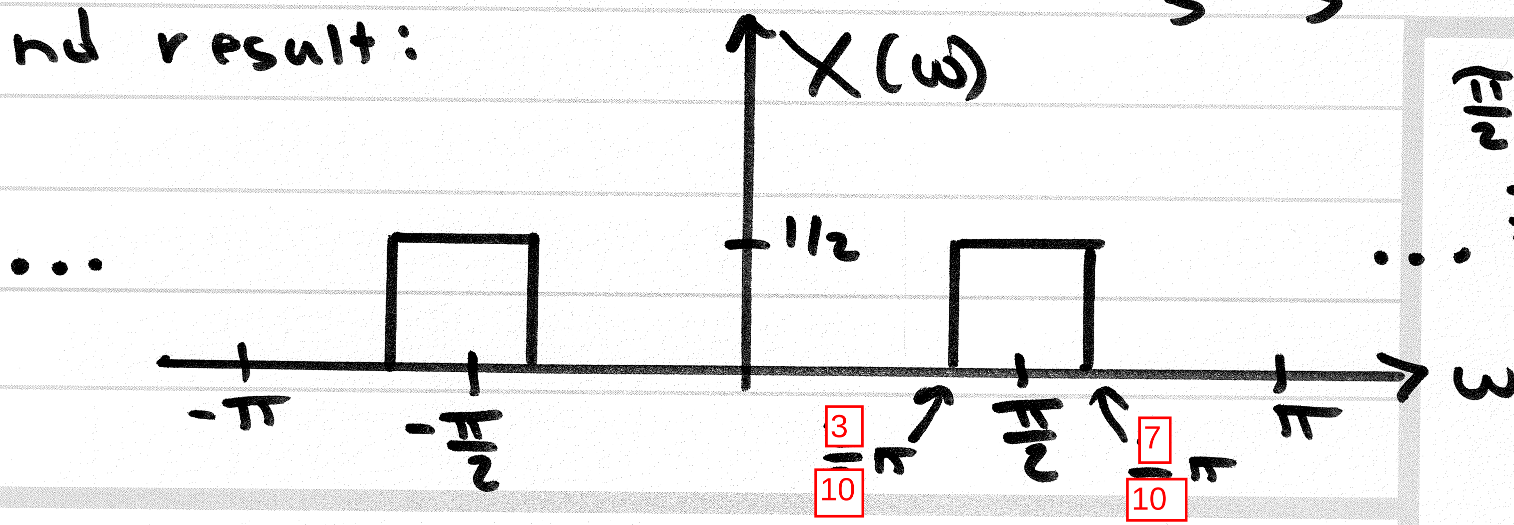
$$X(\omega) = \frac{1}{2}\tilde{X}\left(\omega - \frac{\pi}{2}\right) + \frac{1}{2}\tilde{X}\left(\omega + \frac{\pi}{2}\right)$$

where:

$$\frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n} \stackrel{\mathcal{F}}{\longleftrightarrow}$$



End result:



$$\frac{\pi}{2} - \frac{\pi}{5} = \frac{3}{10}\pi$$

$$\frac{\pi}{2} + \frac{\pi}{5} = \frac{7}{10}\pi$$