• Chap. 7 Sampling Theory

• Basis for the digital revolution

• Theoretical foundations for practical A/D converters and D/A converters

• Initially address the fundamental question of how fast do we have to sample a signal in order to retain full information about the signal

• How do we reconstruct the entire analog signal only knowing its values at equi-spaced instants in time?

- If we multiply $x(t)$ by a periodic train of Dirac Delta functions, we pick off the values at equi-spaced instants in time:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

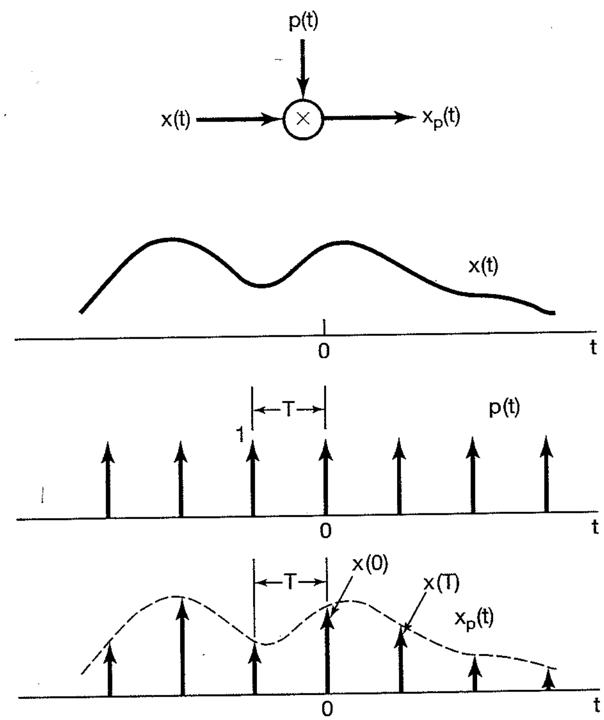


Figure 7.2

$$x_p(t) = x(t)\,p(t) = x(t)\sum_{n}\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

- To analyze in the frequency domain, recall:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{\mathcal{F}} P(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

Ex. 4.8

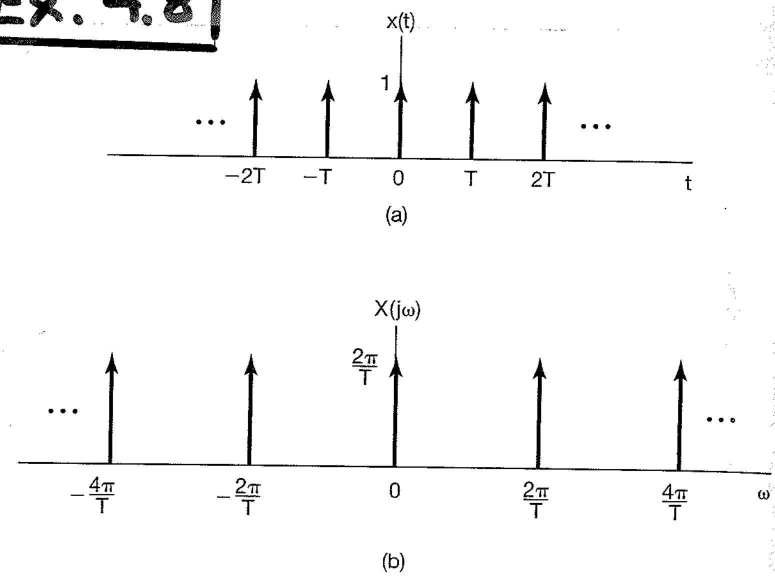


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

- Also, multiplication property of FT dictates:

$$x_p(t) = x(t)\,p(t) \xleftrightarrow{\mathcal{F}} X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

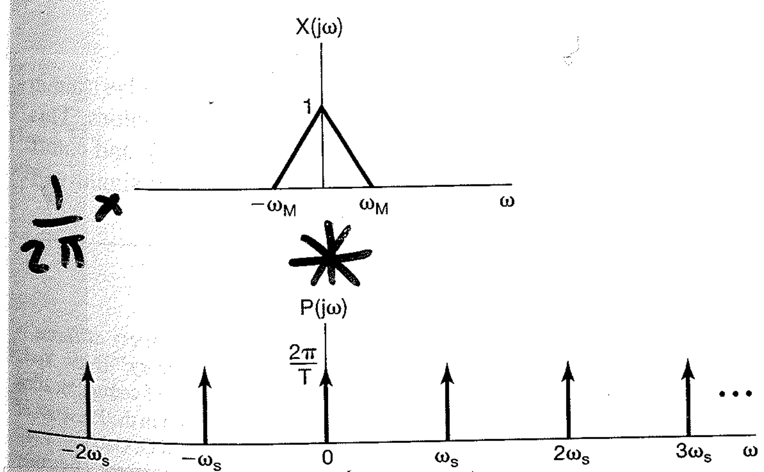
- $$X_p(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega) * \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T_s}\right)$$

$\omega_s > 2\omega_M$

$\frac{1}{2\pi} \times$



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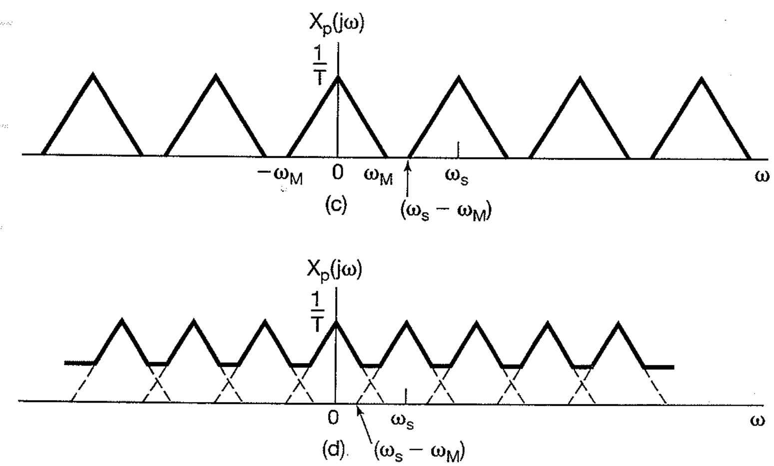


$\omega_s < 2\omega_M$

$\omega_s = \frac{2\pi}{T_s}$ $\qquad \frac{1}{T_s}$ = sampling rate = $\frac{\text{samples}}{\text{sec}}$

ω_M = max frequency in $x(t)$ = bandwidth

- If $\omega_s - \omega_M > \omega_M \Rightarrow \omega_s > 2\omega_M$

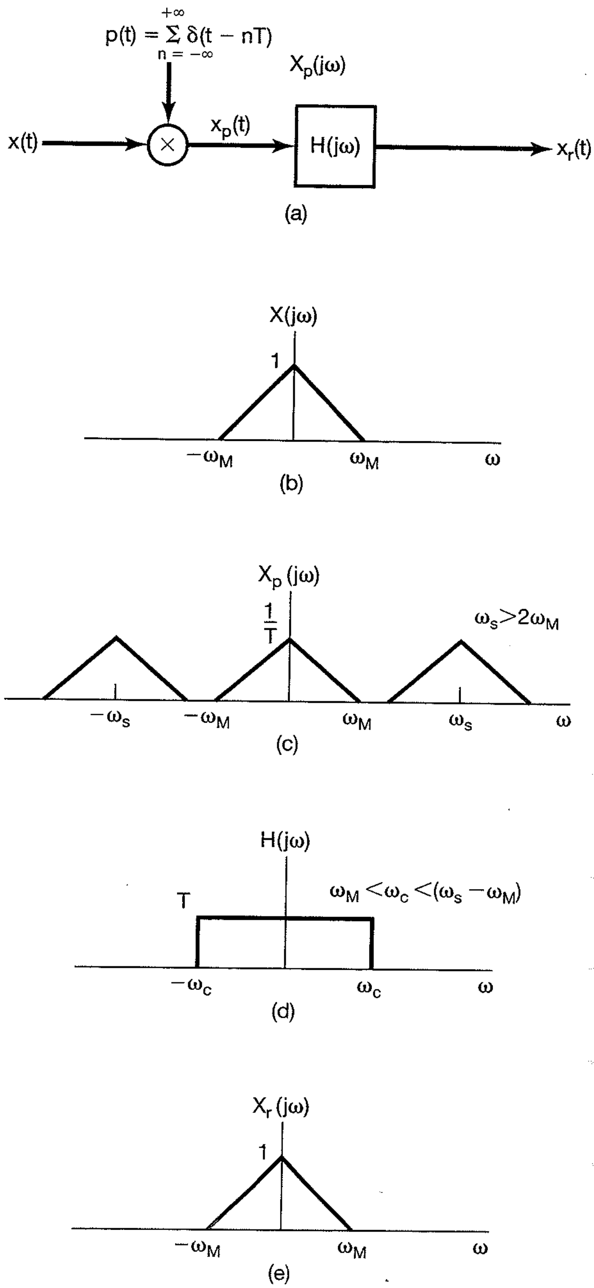
no overlap amongst the spectral replications

$\Rightarrow$ no aliasing

- $2\omega_M$: Nyquist rate

- If $\omega_s > 2\omega_M$, then can recover original signal by lowpass filtering to pass only the original spectrum centered at $\omega = 0$ and reject all the other replicas

- In practice, sample greater than Nyquist rate so that there is a gap between the replica centered at $\omega_s = \frac{2\pi}{T_s}$ and the original spectrum centered at $\omega = 0$, so practical lowpass can roll-off from 1 to 0 (or close to 0 in practice)



- Initially consider Ideal LPF with cut-off at $\omega_c = \frac{\omega_s}{2}$

$$h_{LP}(t) = T_s \frac{\sin\left(\frac{\omega_s}{2} t\right)}{\pi t}$$

$$= T_s \frac{\sin\left(\frac{\pi}{T_s} t\right)}{\pi t}$$

- reconstructed signal:

$$x_r(t) = x_p(t) * h_{LP}(t)$$

$$= \left\{ \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \right\} * h_{LP}(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, h_{LP}(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left(\frac{\pi}{T_s}(t - nT_s)\right)}{\frac{\pi}{T_s}(t - nT_s)}$$

• Thus, if $\omega_s = \frac{2\pi}{T_s} > 2\omega_M$ OR $F_s = \frac{1}{T_s} > 2 f_{max}$

Hz

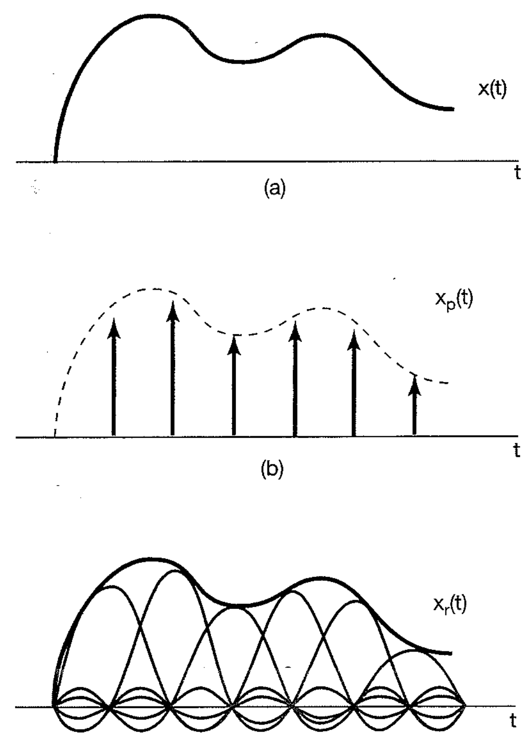
then:

$$x(t) = x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left(\frac{\pi}{T_s}(t - nT_s)\right)}{\frac{\pi}{T_s}(t - nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{\sin\left(\frac{\pi}{T_s}(t - nT_s)\right)}{\frac{\pi}{T_s}(t - nT_s)}$$

just numbers = values of analog signal at equi-spaced instants in time

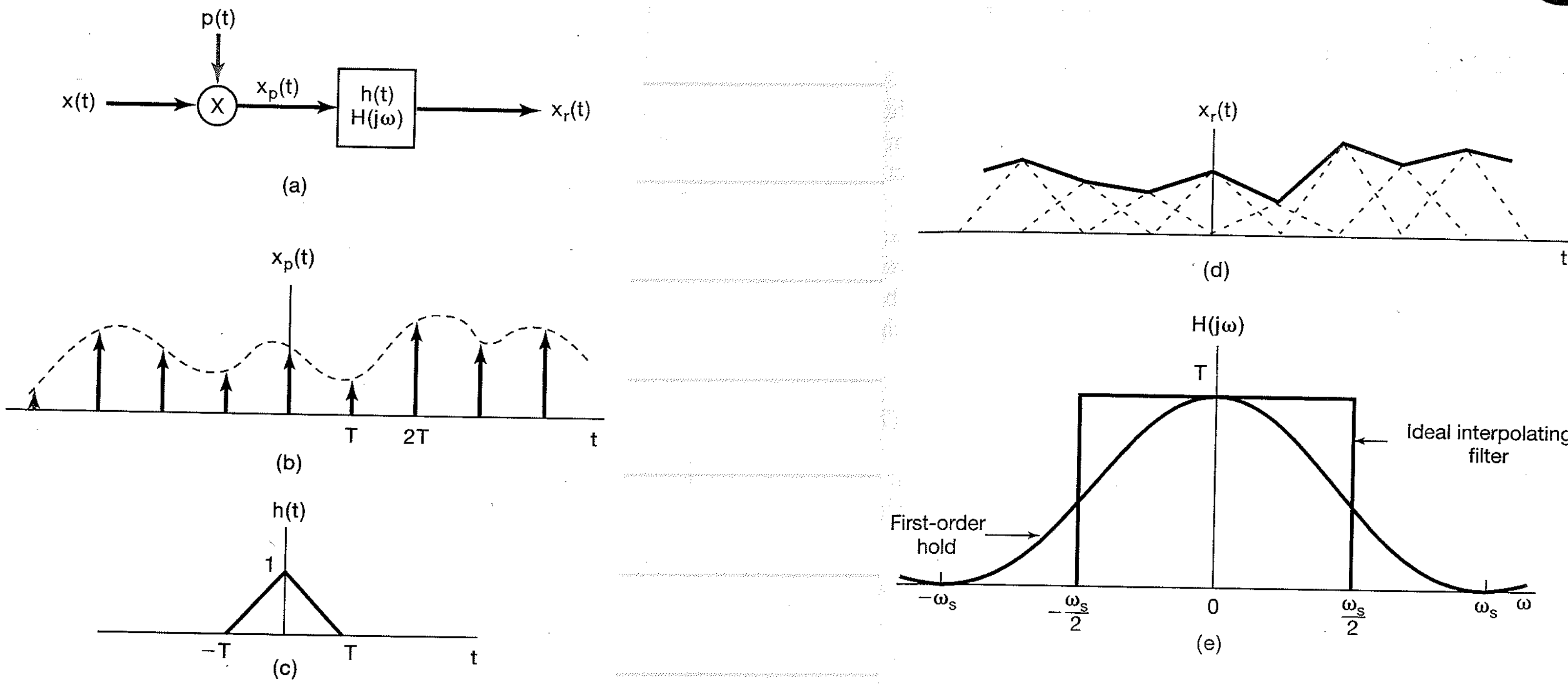
sinc functions = interpolating functions



(a)
(b)

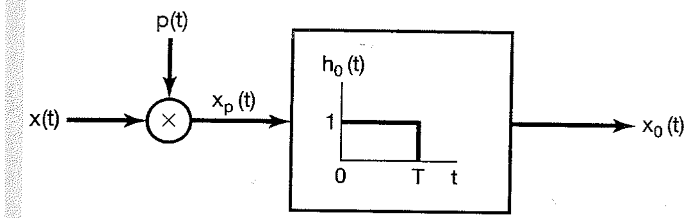
• What about simpler interpolating functions?

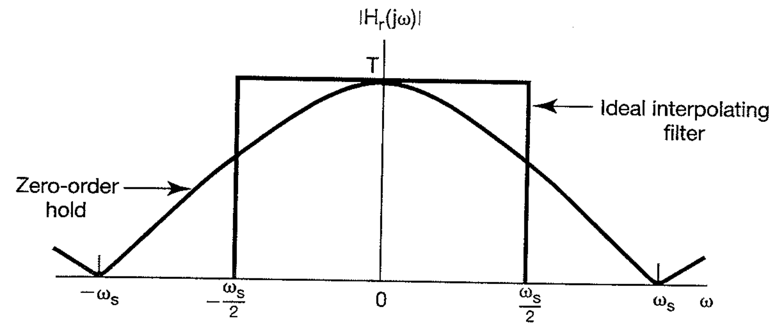
• for example: linear interpolation

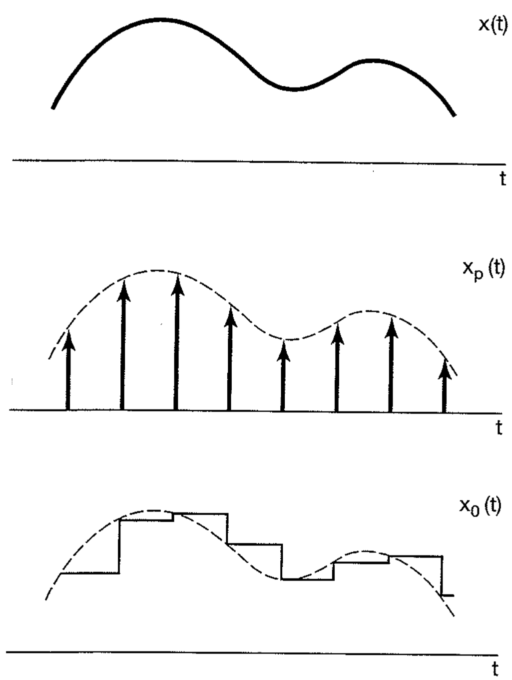


• Will work well if the initial sampling rate was significantly greater than the Nyquist rate

- Simplest scheme: Zero-Order Hold (ZOH)
- charge a capacitor to current sample voltage value and hold constant 'til next sample time







- works well if sampling rate is substantially greater than Nyquist rate
- Sect. 7.5.2 describes a way to increase the sampling rate digitally just prior to D/A conversion
- most often done in practice