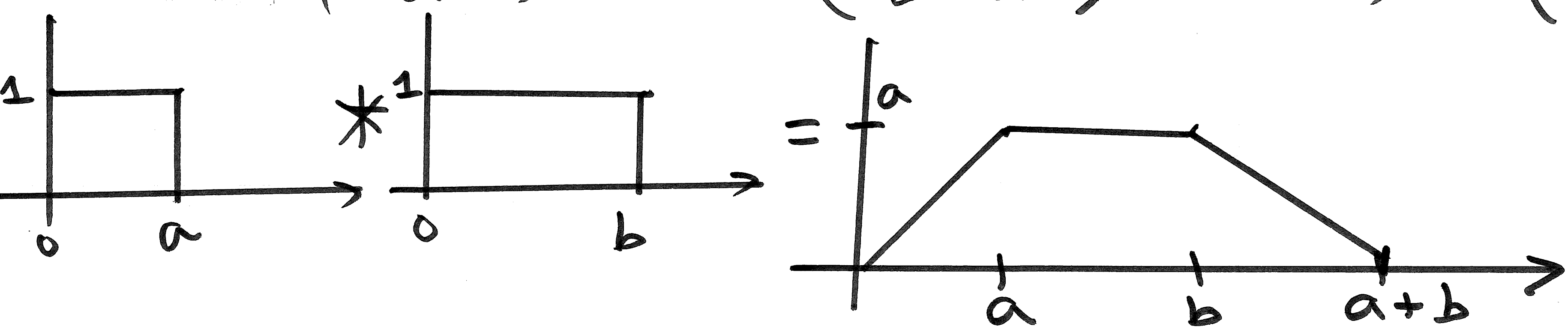
Some Basic Convolution Results

$$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}\left\{e^{at} - e^{bt}\right\}u(t)$$

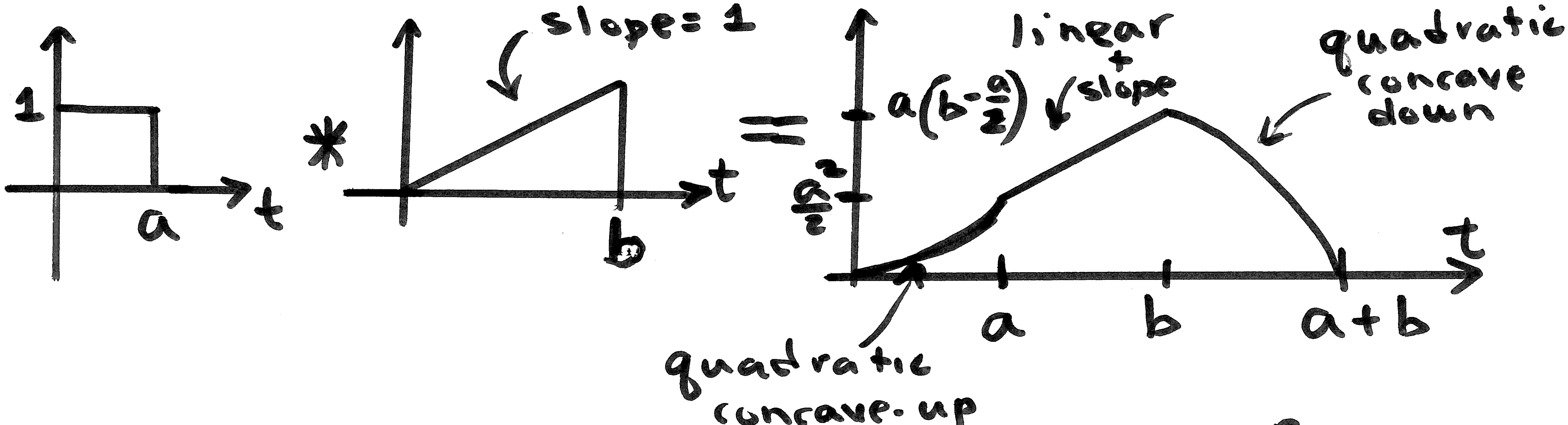
a, b complex-valued or real-valued or zero (but not $a=b$)

$$\mathrm{rect}\left(\frac{t-\frac{a}{2}}{a}\right) * \mathrm{rect}\left(\frac{t-\frac{b}{2}}{b}\right) =$$

$$= t\,\mathrm{rect}\left(\frac{t-\frac{a}{2}}{a}\right) + a\,\mathrm{rect}\left(\frac{t-\frac{a+b}{2}}{b-a}\right) + (b+a-t)\,\mathrm{rect}\left(\frac{t-(b+\frac{a}{2})}{a}\right)$$



- ## Ramp-Up Linearly - Triangle Conv. Result



$$(u(t)-u(t-a)) * t\,(u(t)-u(t-b))$$

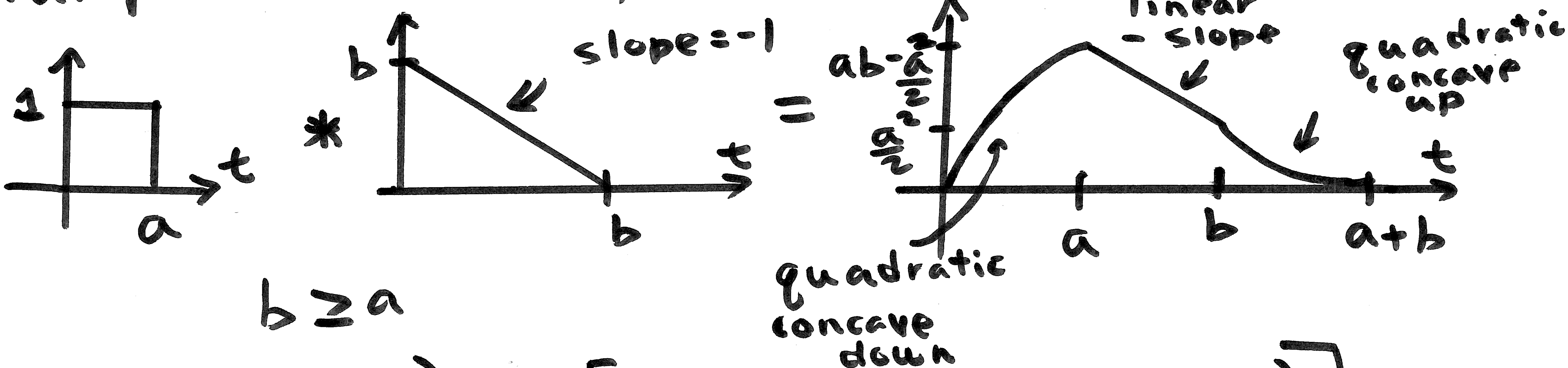
for $b \geq a$

$$= \frac{t^2}{2}\,(u(t)-u(t-a))$$

$$+ \left(at - \frac{a^2}{2}\right)(u(t-a)-u(t-b))$$

$$+ \left(-\frac{t^2}{2} + at + \frac{b^2-a^2}{2}\right)(u(t-b)-u(t-(a+b)))$$

~~Ramp Down~~ Linearly - Triangle Conv. Result



$b \geq a$

$$(u(t)-u(t-a)) * \left[-(t-b)\left(u(t)-u(t-b)\right)\right] =$$

$$= \left(-\frac{t^2}{2}+bt\right)\left(u(t)-u(t-a)\right)$$

$$+\left(-at+\frac{a^2}{2}+ab\right)\left(u(t-a)-u(t-b)\right)$$

$$+\left(\frac{t^2}{2}-(a+b)t+\frac{(a+b)^2}{2}\right)\left(u(t-b)-u(t-(a+b))\right)$$

The basic convolution result below is derived later in this same set of notes

$$e^{at}\{u(t)-u(t-T_a)\} * e^{bt}\{u(t)-u(t-T_b)\}$$

$$= \tilde{y}(t) - e^{bT_b}\tilde{y}(t-T_b) - e^{aT_a}\tilde{y}(t-T_a)$$

for $0 < t < (T_a+T_b)$

$= 0$ for $t<0$ and $t > (T_a+T_b)$

• where: $\tilde{y}(t) = \left\{\dfrac{e^{at}-e^{bt}}{a-b}\right\} u(t)$

Summary of Some Key Results Related to Convolution

If: $y(t) = x(t) * h(t)$

Then: $a\,x(t-t_1) * b\,h(t-t_2)$

$$= ab\, y(t-(t_1+t_2))$$

If: $y_i(t) = x_i(t) * h(t), \quad i = 1, 2$

$$\big(a\,x_1(t-t_1) + b\,x_2(t-t_2)\big) * h(t)$$

$$= a\,y_1(t-t_1) + b\,y_2(t-t_2)$$

$$\big(a\,x_1(t-t_1) + b\,x_2(t-t_2)\big) * h(t-t_0) =$$

$$= a\,y_1(t-(t_1+t_0)) + b\,y_2(t-(t_2+t_0))$$

- Additional Properties of Delta Function:

$$\boxed{x(t) * \delta(t) = x(t)} \qquad \boxed{x(t) * \delta(t-t_0) = x(t-t_0)}$$

Proof: $x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(\tau - t_0)\, x(t-\tau)\, d\tau$

sifting property of Dirac Delta function

- sifting property dictates $= \int_{-\infty}^{\infty} \delta(\tau - t_0)\, x(t-t_0)\, d\tau$

- $x(t-t_0)$ does not depend on integration variable τ $= x(t-t_0) \int_{-\infty}^{\infty} \delta(\tau - t_0)\, d\tau$

- area under Delta Function is unity $= x(t-t_0)$

Can easily show: $\delta(t-t_1) * \delta(t-t_2) = \delta\big(t-(t_1+t_2)\big)$

Further Props. of Convolution

1. Commutative: $x(t) * h(t) = h(t) * x(t)$

2. Associative: $\left(x(t) * h_1(t)\right) * h_2(t)$

$= (x(t) * h_1(t)) * h_2(t) = x(t) * \left(h_2(t) * h_1(t)\right)$

$\Rightarrow$ Two LTI systems in series:

1. order doesn't matter
2. can be replaced by single LTI system
   $h(t) = h_1(t) * h_2(t)$

3. Distributive:

$x(t) * \left(h_1(t) + h_2(t)\right)$

$= x(t) * h_1(t) + x(t) * h_2(t)$

$\Rightarrow$ Two LTI systems in parallel can be replaced by single system: $h(t) = h_1(t) + h_2(t)$

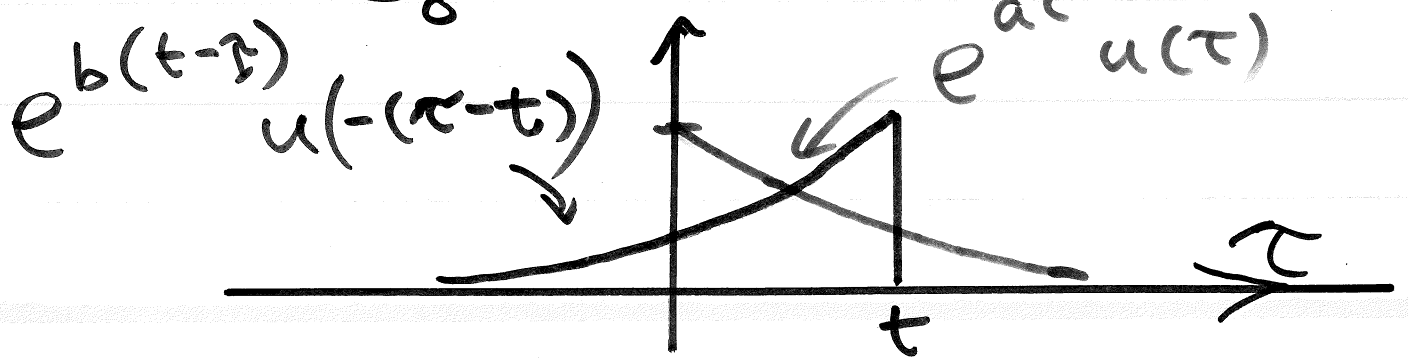
Prob. 2.22 (a) $(a=-\alpha,\ b=-\beta)$

$$e^{at}u(t) * e^{bt}u(t) = ? \quad \text{for } b \neq a$$

$$y(t) = \int_{-\infty}^{\infty} e^{a\tau} u(\tau)\, e^{b(t-\tau)} u(t-\tau)\, d\tau$$

$$= e^{bt}\int_{-\infty}^{\infty} e^{(a-b)\tau} u(\tau)\, u(-(\tau-t))\, d\tau$$

$$= e^{bt}\int_{0}^{t} e^{(a-b)\tau}\, d\tau \quad u(t)$$



$$y(t) = e^{bt}\left\{\left.\frac{1}{a-b}e^{(a-b)\tau}\right]_0^t\right\}$$

$$= \frac{e^{bt}}{a-b}\left\{e^{(a-b)t} - e^{0}\right\}$$

$$= \left\{\frac{e^{at} - e^{bt}}{a-b}\right\}u(t) \qquad = y(t)$$

$$= e^{at}u(t) * e^{bt}u(t) \qquad a \neq b$$

$$= y(t) \quad (\text{notation}) = y(t)$$

- Suppose we truncate the exponentials to finite length

$$z(t) = e^{at}\{u(t) - u(t-T_a)\} * e^{bt}\{u(t)-u(t-T_b)\}$$

Use FOIL (linearity) + time-invariance

$$= \{e^{at}u(t) - e^{aT_a}e^{a(t-T_a)}u(t-T_a)\} *$$

$$\{e^{bt}u(t) - e^{bT_b}e^{b(t-T_b)}u(t-T_b)\}$$

$$= y(t) - e^{bT_b}y(t-T_b) - e^{aT_a}y(t-T_a)$$

$$+ e^{aT_a}e^{bT_b}y(t-(T_a+T_b))$$

- Suppose $T_a = T_b = T$
- and, recall: $y(t) = \left\{ \dfrac{e^{at} - e^{bt}}{a-b} \right\} u(t)$

- for $0 < t < T$: $\quad z(t) = y(t)$

- for $T < t < 2T$:

$$z(t) = y(t) - e^{bT} y(t-T) - e^{aT} y(t-T)$$
$$= y(t) - \left(e^{bT} + e^{aT}\right) y(t-T)$$

- for $t > 2T$: $\quad z(t) = 0$

Prove:

$$y(t) - \left(e^{bT} + e^{aT}\right) y(t-T) + e^{aT} e^{bT} y(t-2T) = 0$$
$$= \text{zero!!}$$

Note: Result below can be used for convolving exponential with rectangle by setting b=0.

What about

$$e^{at}u(t) * e^{bt}\{u(t)-u(t-T)\} = ?$$

- Distributive Prop. of Convolution:

$$= e^{at}u(t) * e^{bt}u(t) - e^{at}u(t) * e^{bt}u(t-T)$$

$$= e^{at}u(t) * e^{bt}u(t) - e^{bT}\left(e^{at}u(t) * e^{b(t-T)}u(t-T)\right)$$

$$= \frac{1}{a-b}\{e^{at} - e^{bt}\}u(t)$$

$$- \frac{e^{bT}}{a-b}\{e^{a(t-T)} - e^{b(t-T)}\}u(t-T)$$

Trick: Recall: $\delta(t) = \frac{d}{dt}\{u(t)\}$

Thus:

$$u(t) \longrightarrow \boxed{\begin{matrix}\text{Differentiator} \\ h_d(t)\end{matrix}} \longrightarrow \delta(t)$$

- $\delta(t) = h_d(t) * u(t)$
- What is impulse response of a differentiator?
- You don't want to know :) (A "doublet")
- So, let's write as: $\delta(t) = \left\{\frac{d}{dt}\right\} * u(t)$
- <u>Since</u> $x(t) = x(t) * \delta(t)$

$$y(t) = x(t) * \left\{\left\{\frac{d}{dt}\right\} * u(t)\right\} * h(t)$$

$$= \frac{dx(t)}{dt} * u(t) * h(t)$$

$$= x(t) * u(t) * \frac{dh(t)}{dt}$$

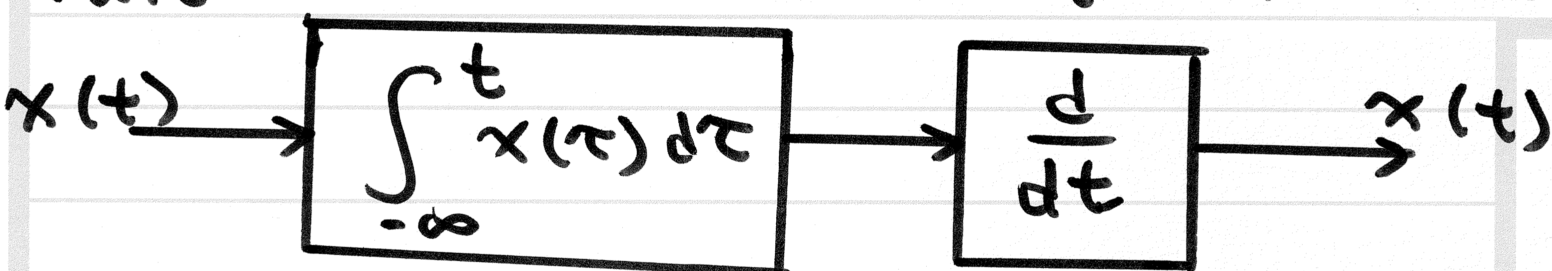
$$= x(t) * \frac{dh(t)}{dt} * u(t)$$

• Similarly for a CT LTI System with impulse response $h(t)$, if the inverse system exists, it must satisfy:

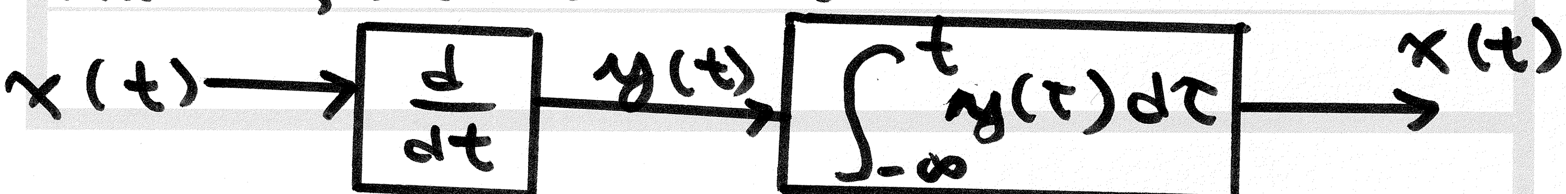
$$h(t) * h_I(t) = \delta(t)$$

where $h_I(t)$ is the impulse response of the inverse system

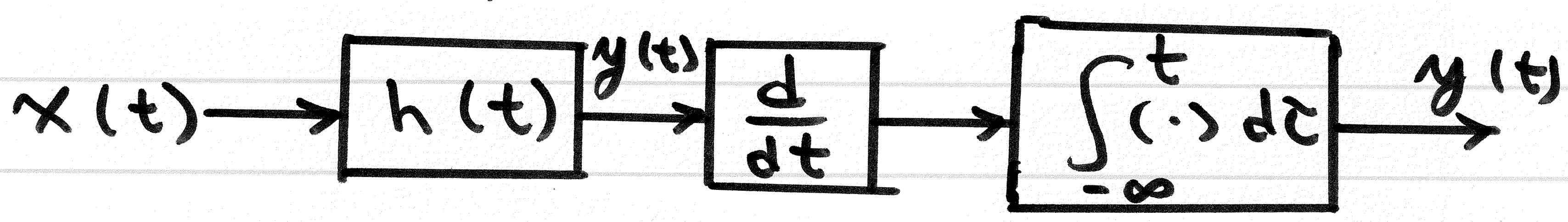
• Although it's peculiar to talk about the impulse response of a differentiator, one can use Leibniz's rule to show the following inverse system pair



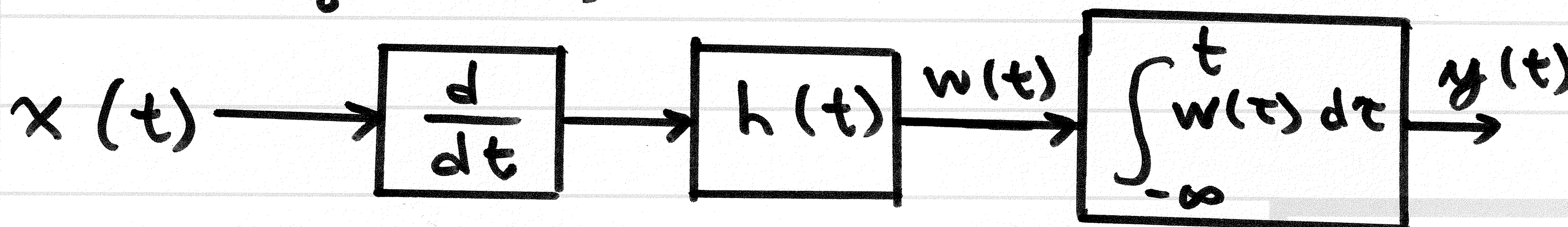
• since the order of LTI systems in series doesn't matter, we also have:



- Consider an arbitrary LTI system whose impulse response is $h(t)$. We have:



- Since order of LTI Systems in series is inconsequential, we have:



- This "trick" was sometimes used in ECE 202 to simplify the computation of $x(t) * h(t)$ if $x(t)$ was piecewise flat or piecewise linear

CT LTI System Examples

System 1

$$y(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} x(\tau)\, d\tau$$

Let: $x(t) = \delta(t) \Rightarrow y(t) = h(t)$

$$h(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} \delta(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} e^{-a(t-0)} \delta(\tau)\, d\tau$$

Sifting Property of Dirac-Delta Function

$$= e^{-at} \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$= e^{-at} u(t)$$

CT LTI System Examples

System 2:

$$y(t) = -\frac{1}{T}\int_{t-T}^{t}(t-\tau-T)\,x(\tau)\,d\tau$$

Let $x(t) = \delta(t) \Rightarrow y(t) = h(t)$

$$h(t) = -\frac{1}{T}\int_{t-T}^{t}(t-\tau-T)\,\delta(\tau)\,d\tau$$

$$= -\frac{1}{T}\int_{t-T}^{t}(t-0-T)\,\delta(\tau)\,d\tau$$

$$= -\frac{1}{T}(t-T)\left\{\int_{t-T}^{t}\delta(\tau)\,d\tau\right\}$$

$t>0$ & $t-T<0$

$t<T$

so $= 1$ if $0<t<T$

$= 0$ otherwise

$$= -\frac{1}{T}(t-T)\left\{u(t)-u(t-T)\right\}$$

$\mathrm{rect}\left(\frac{t-\frac{T}{2}}{T}\right)$