All-Pass Filters

Note wrt All-Pass Filters.

- mathematical preliminary

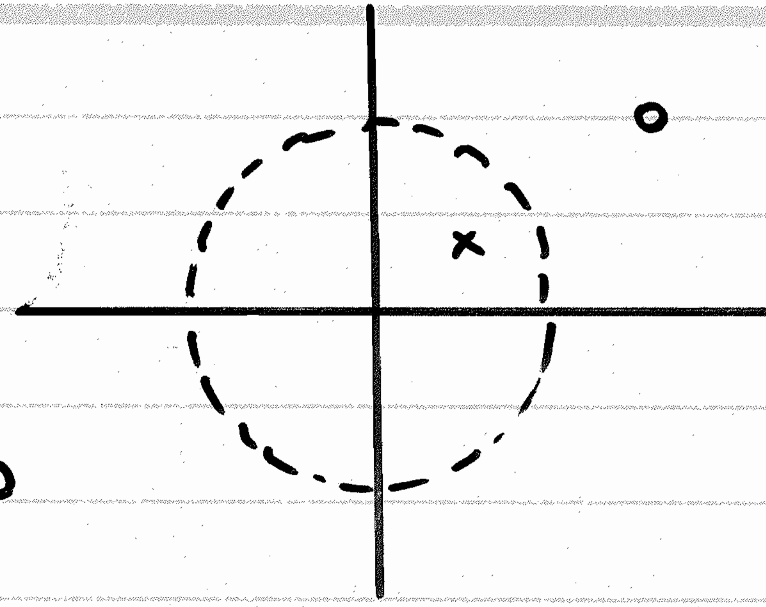
- let $c = a + jb = |c| e^{j\angle c}$

- note: $\dfrac{c}{c^*} = \dfrac{|c| e^{j\angle c}}{|c| e^{-j\angle c}} = 1\, e^{j 2\angle c}$

- THUS: $\left|\dfrac{c}{c^*}\right| = 1$

- Now, consider system with single pole at $z = p$ and a single zero at $z = \dfrac{1}{p^*}$

$$H(z) = G \frac{\left(z - \frac{1}{p^*}\right)}{z - p}$$



$$H(\omega) = H(z)\Big|_{z = e^{j\omega}}$$

$$H(\omega) = G \frac{\left(e^{j\omega} - \frac{1}{p^*}\right)}{e^{j\omega} - p} = \frac{G \frac{1}{p^*} e^{+j\omega}\left(p^* - e^{-j\omega}\right)}{-\left(p - e^{j\omega}\right)}$$

$$= -\frac{G}{p^*} e^{j\omega} \frac{c}{c^*} \qquad \text{where: } c = p - e^{j\omega}$$

THUS: $|H(\omega)| = \left|\frac{G}{p^*}\right| = \frac{|G|}{|p|}$ } does not depend on ω

$\Rightarrow$ ALL PASS!

- An all-pass filter can be used to stabilize an unstable system without affecting the magnitude of the frequency response
- Suppose there is a pole at p outside unit circle $\Rightarrow$ $1/p^*$ is inside unit circle

$$\underbrace{\frac{H'(z)}{(z-p)}}_{\text{everything but } (z-p)} \times \frac{(z-p)}{(z-1/p^*)}$$

zero-pole cancellation

now have new pole at $z = 1/p^*$ inside unit circle

$\Rightarrow$ magnitude is unaffected

• Consider real-valued all-pass filter with single-pole:

$$H(z) = p\,\frac{z - 1/p}{z - p} \Rightarrow |H(\omega)| = 1 \quad \forall\, \omega$$

$$r_{hh}[\ell] = h[\ell] * h[-\ell] \overset{DTFT}{\longleftrightarrow} |H(\omega)|^2 = 1$$

$$= \delta[\ell]$$

• What is $h[n]$?

$$H(z) = p\,\frac{z}{z-p} - z^{-1}\frac{z}{z-p}$$

• Thus:

$$h[n] = p\; p^n u[n] - p^{n-1} u[n-1]$$

$$= p\; p^n u[n] - \frac{1}{p}\; p^n\; u[n]$$

- Some algebraic manipulation:

$$h[n] = \alpha\,\delta[n] + \left(p - \frac{1}{p}\right) p^n u[n]$$

- where: $p = \alpha + p - \frac{1}{p} \Rightarrow \alpha = \frac{1}{p}$

$$h[n] = \frac{1}{p}\delta[n] + \left(\frac{p^2-1}{p}\right) p^n u[n]$$

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let: $x[n] = \frac{1}{p}\left\{\delta[n] + (p^2-1)\,p^n u[n]\right\}$

- One can verify that:

$$r_{xx}[\ell] = x[\ell] * x[-\ell] = \delta[\ell]$$

- note: $x[n]$ is <u>not</u> constant modulus

- also, recall:

$$x[n] = p^n u[n] \Rightarrow r_{xx}[\ell] = \frac{1}{1-p^2}\, p^{|\ell|}$$