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Discrete Nonlinear Optimal Control of S/C Formations Near the L_1 and L_2 Points of the Sun-Earth/Moon System

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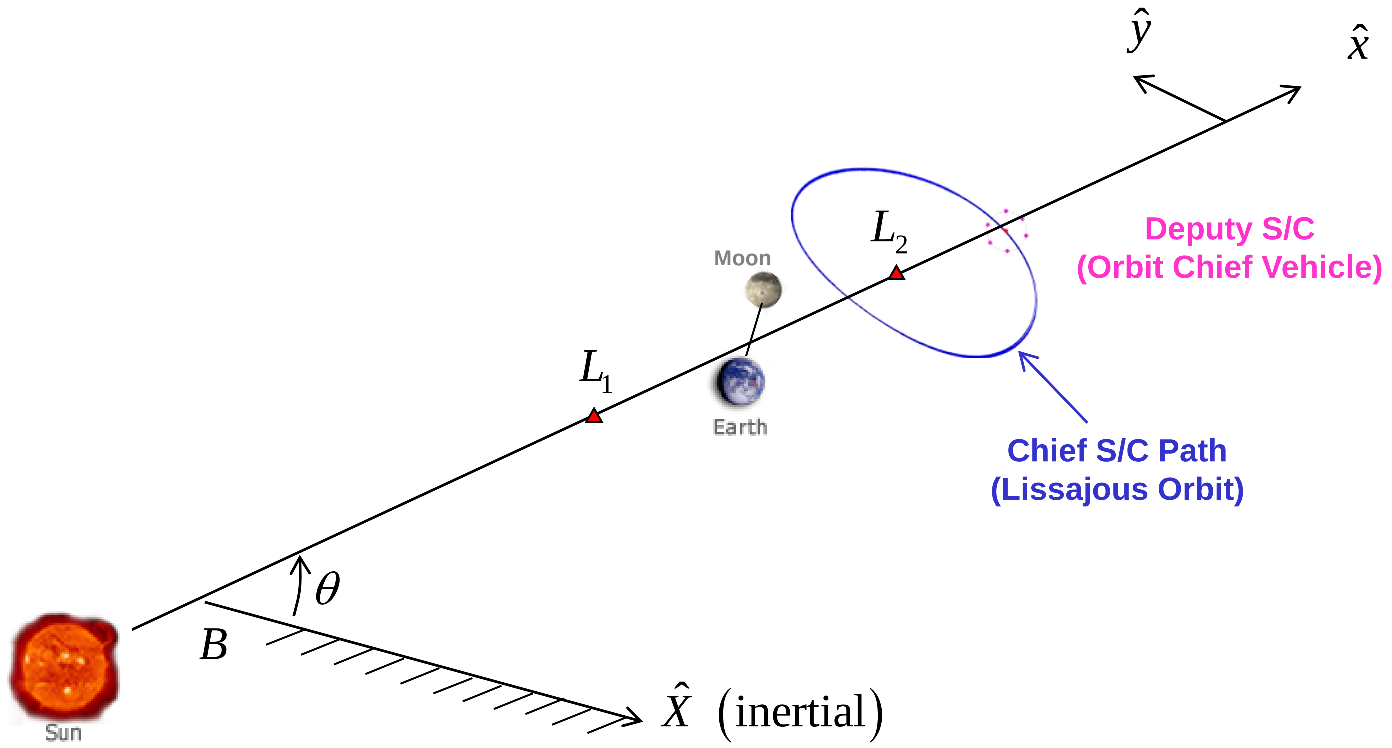
Formations Near the Libration Points



EPHEM = Sun + Earth + Moon Motion From Ephemeris w/ SRP
CR3BP = Sun + Earth/Moon barycenter Motion Assumed Circular w/o SRP

Formation Keeping via Nonlinear Optimal Control

- Incorporate nonlinearities into control design process
- Allows for the addition of control and path constraints
  - Upper and lower bounds on thrust output
  - Specifications on relative path error
  - Allow for thruster on-off times while min. the impact on the path
- Min. # of assumptions → better assessment of feasibility

Optimal Control Solution

- Method #1: Partial Discretization
  - Divide Trajectory into Segments and Nodes
  - Numerically integrate node states
  - Impulsive Control at Nodes (or Constant Thrust Between Nodes)
  - Numerically integrated gradients
  - Solve Using Subspace Trust Region Method
- Method #2: Transcription and Nonlinear Programming
  - Divide Trajectory Into Segments and Nodes
  - Solve using Sparse Optimal Control Software (SOCS)
    - Use Hermite-Simpson discretization (others available)
    - Jacobian and Hessian computed via Sparse Finite Differencing.
    - Estimate cost index to second order
    - Use SQP algorithm

Identification of Startup Solution

- Possible Startup Solution Options
  - Non-Natural Arcs → IFL/OFL Nonlinear Control
    - Specify some nominal motion
    - Apply IFL/OFL control to achieve desired nominal
    - Use results as initial guess to optimal control process with $\bar{u}(t) \neq 0$
  - Natural Arcs → Floquet Analysis of Chief S/C Linearized Equations
    - Deputy dynamics modeled as a perturbation relative to chief path
    - Floquet controller applied to establish natural relative formation
    - Transition into NL system via 2-level corrector
    - Use results as initial guess to optimal control process with $\bar{u}(t) = 0$

Method #1: Optimal Control by Partial Discretization

$$\min J = \phi\left(\bar{x}_N\right) + \sum_{j=0}^{N-1} L\left(t_j, \bar{x}_j, \bar{u}_j\right) = \phi\left(\bar{x}_N\right) + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \tilde{L}\left(t, \bar{x}, \bar{u}\right) dt$$

Subject to:

$$\bar{x}_{j+1} = \bar{F}\left(t_j, \bar{x}_j, \bar{u}_j\right); \text{ Subject to } \bar{x}\left(0\right) = \bar{x}_0 = \bar{x}_i$$

Equivalent Representation as Augmented Nonlinear System:

$$\min \tilde{J} = \phi\left(\bar{x}_N\right) + x_{n+1}\left(t_N\right) = \tilde{\phi}\left(\tilde{x}_N\right)$$

$$\tilde{x}_{j+1} = \begin{bmatrix} \bar{x}_{j+1} \\ x_{n+1}\left(t_{j+1}\right) \end{bmatrix} = \begin{bmatrix} \bar{F}\left(t_j, \bar{x}_j, \bar{u}_j\right) \\ x_{n+1}\left(t_j\right) + L\left(t_j, \bar{x}_j, \bar{u}_j\right) \end{bmatrix} = \tilde{F}\left(t_j, \tilde{x}_j, \bar{u}_j\right);$$

$$\text{Subject to } \tilde{x}_0 = \begin{bmatrix} \bar{x}_0 \\ 0 \end{bmatrix}$$

Euler-Lagrange Optimality Conditions (Based on Calculus of Variations)

$$H_j = \tilde{\lambda}_{j+1}^T \tilde{F}\left(t_j, \overline{x}_j, \overline{u}_j\right)$$

Condition #1: $\tilde{\lambda}_j^T = \dfrac{\partial H_j}{\partial \tilde{x}_j} = \tilde{\lambda}_{j+1}^T \dfrac{\partial \tilde{F}_j}{\partial \tilde{x}_j} \quad \rightarrow \quad \tilde{\lambda}_N^T = \begin{bmatrix} \dfrac{\partial \phi\left(\overline{x}_N\right)}{\partial \overline{x}_N} & 1 \end{bmatrix}$

Condition #2: $\overline{0} = \dfrac{\partial H_j}{\partial \overline{u}_j} = \tilde{\lambda}_{j+1}^T \dfrac{\partial \tilde{F}_j}{\partial \overline{u}_j}; \quad j = 0, \ldots, N-1$

Identify $\dfrac{\partial \tilde{F}_j}{\partial \tilde{x}_j}$ and $\dfrac{\partial \tilde{F}_j}{\partial \overline{u}_j}$ from augmented linear system.

Identification of Gradients From the Augmented Linearized Model

Nonlinear System:

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} \bar{f}(t,\bar{x},\bar{u}) \\ \tilde{L}(t,\bar{x},\bar{u}) \end{bmatrix}; \quad \begin{bmatrix} \bar{x}(0) \\ x_{n+1}(0) \end{bmatrix} = \begin{bmatrix} \bar{x}_0 \\ 0 \end{bmatrix}$$

Linear System:

$$\delta\dot{\tilde{x}}(t) = \tilde{A}(t)\,\delta\tilde{x}(t) + \tilde{B}(t)\,\delta\bar{u}(t)$$

$$\tilde{A}(t) = \begin{bmatrix} A_d(t) & \bar{0} \\ \dfrac{\partial\tilde{L}}{\partial\bar{x}} & \bar{0} \end{bmatrix} \qquad \tilde{B}(t) = \begin{bmatrix} 0_3 \\ I_3 \\ \bar{0}^T \end{bmatrix}$$

Solution to Linearized Equations

$$\delta\tilde{x}(t) = \tilde{\Phi}(t,t_0)\,\delta\tilde{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)\,\tilde{B}(\tau)\,\delta\overline{u}(\tau)\,d\tau$$

$$\dot{\tilde{\Phi}}(t,t_0) = \tilde{A}(t)\,\tilde{\Phi}(t,t_0); \quad \tilde{\Phi}(t_0,t_0) = I_7$$

Relation to Gradients in E-L Optimality Conditions:

$$\delta\tilde{x}_{j+1} = \underbrace{\tilde{\Phi}(t_{j+1},t_j)}_{\dfrac{\partial\tilde{F}}{\partial\tilde{x}_j}}\,\delta\tilde{x}_j + \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},\tau)\,\tilde{B}(\tau)\,\delta\overline{u}(\tau)\,d\tau$$

Control Gradient for Impulsive Control

$$\delta\tilde{x}_{j+1}^{-} = \tilde{\Phi}\left(t_{j+1}, t_j\right)\delta\tilde{x}_j^{+} + \int_{t_j}^{t_{j+1}} \Phi\left(t_{j+1}, \tau\right)\tilde{B}\left(\tau\right)\delta\overline{u}\left(\tau\right)d\tau$$

$$= \tilde{\Phi}\left(t_{j+1}, t_j\right)\left(\delta\tilde{x}_j^{-} + \tilde{B}\Delta\overline{V}_j\right)$$

$$= \tilde{\Phi}\left(t_{j+1}, t_j\right)\delta\tilde{x}_j^{-} + \underbrace{\tilde{\Phi}\left(t_{j+1}, t_j\right)\tilde{B}}_{\frac{\partial\tilde{F}}{\partial\overline{u}_j}}\Delta\overline{V}_j$$

$$\boxed{\frac{\partial\tilde{F}}{\partial\overline{u}_j} = \Phi\left(t_{j+1}, t_j\right)\tilde{B}}$$

Control Gradient for Constant Thrust Arcs

$$\delta\tilde{x}_{j+1} = \tilde{\Phi}\left(t_{j+1}, t_j\right)\delta\tilde{x}_j + \underbrace{\left[\int_{t_j}^{t_{j+1}} \Phi\left(t_{j+1},\tau\right)\tilde{B}\left(\tau\right)d\tau\right]}_{\frac{\partial\tilde{F}}{\partial\bar{u}_j}}\delta\bar{u}_j$$

Only $\Phi\left(\tau, t_j\right)$ available from numerical integration

Use STM properties to rewrite $\frac{\partial\tilde{F}}{\partial\bar{u}}$ in terms of $\Phi\left(\tau, t_j\right)$.

$$\frac{\partial\tilde{F}}{\partial\bar{u}_j} = \Phi\left(t_{j+1}, t_j\right)\left[\int_{t_j}^{t_{j+1}} \Phi\left(\tau, t_j\right)^{-1}\tilde{B}\left(\tau\right)d\tau\right]$$

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{x}_{n+1} \\ \dot{\tilde{\Phi}}\left(t,t_j\right) \\ \dot{\Phi}^*\left(t,t_j\right) \end{bmatrix} = \begin{bmatrix} \bar{f}\left(t,\bar{x},\bar{u}\right) \\ L\left(t,\bar{x},\bar{u}\right) \\ \tilde{A}\left(t\right)\tilde{\Phi}\left(t,t_j\right) \\ \tilde{\Phi}\left(t,t_j\right)^{-1}\tilde{B}\left(\tau\right) \end{bmatrix}$$

Numerical Solution Process

(1) Input $\tilde{x}_0, t_N$, and initial guess for $\overline{u}_i$; $(i = 0,1,\ldots, N-1)$

(2) 1-Scalar Equation to Optimize in $3(N-1)$ Control Variables

Use optimizer to identify optimal $\overline{u}_i$ given $\dfrac{\partial H_i}{\partial \overline{u}_i}$.

During each iteration of the optimizer, the following steps are followed:

(a) Sequence (by numerical integration) $\overline{x}_i$ forward and store; $i = 1,\ldots, N-1$

(b) Evaluate cost functional, $J = \tilde{\phi}(\tilde{x}_N)$

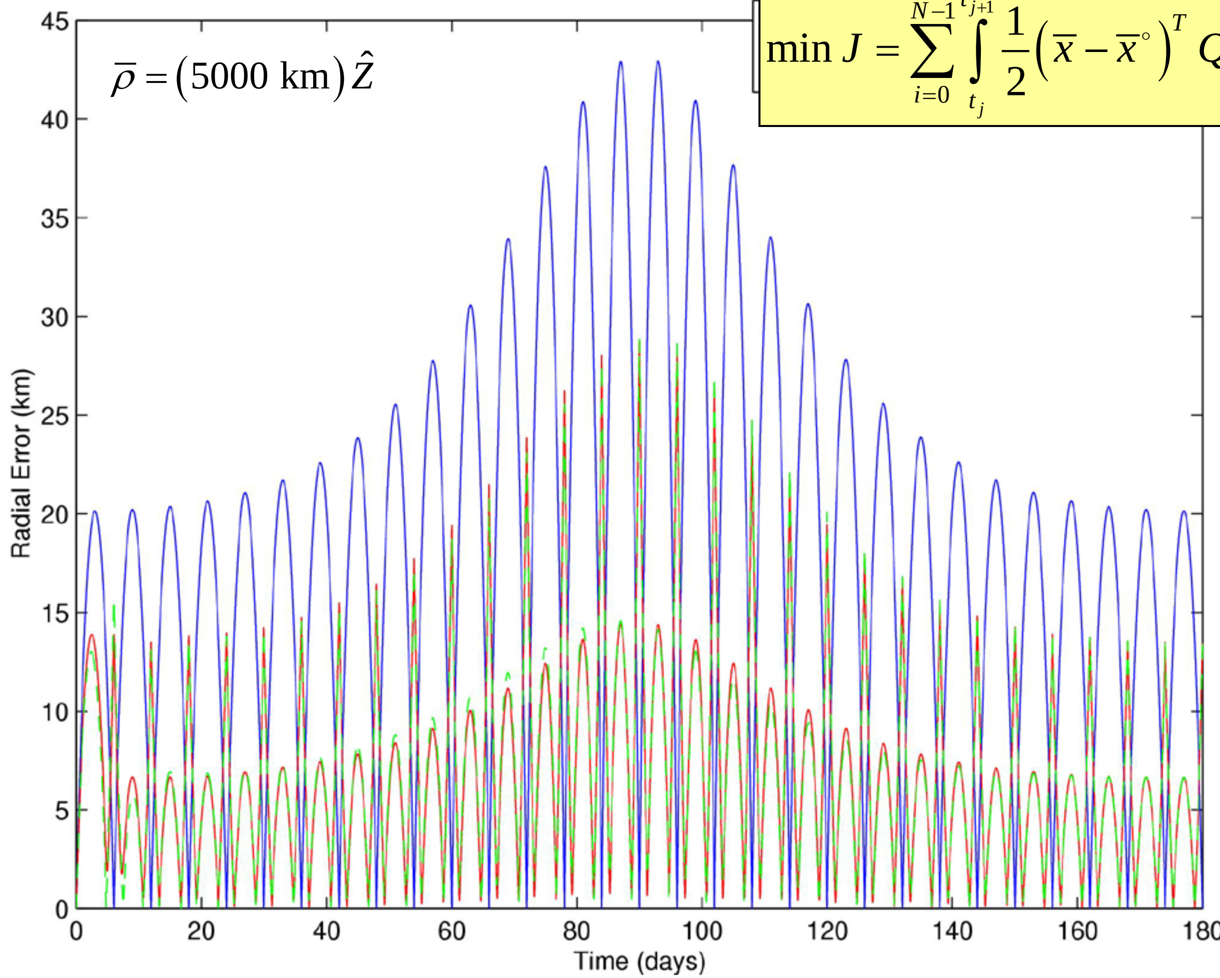
(c) Evaluate $\tilde{\lambda}_N^T = \dfrac{\partial \tilde{\phi}(\tilde{x}_N)}{\partial \tilde{x}_N} = \begin{bmatrix} \dfrac{\partial \phi_N}{\partial \overline{x}_N} & 1 \end{bmatrix}$

(d) Sequence $\tilde{\lambda}_i$ backward and compute the search direction $\dfrac{\partial H_i}{\partial \overline{u}_i}$; $i = N-1,\ldots,1$

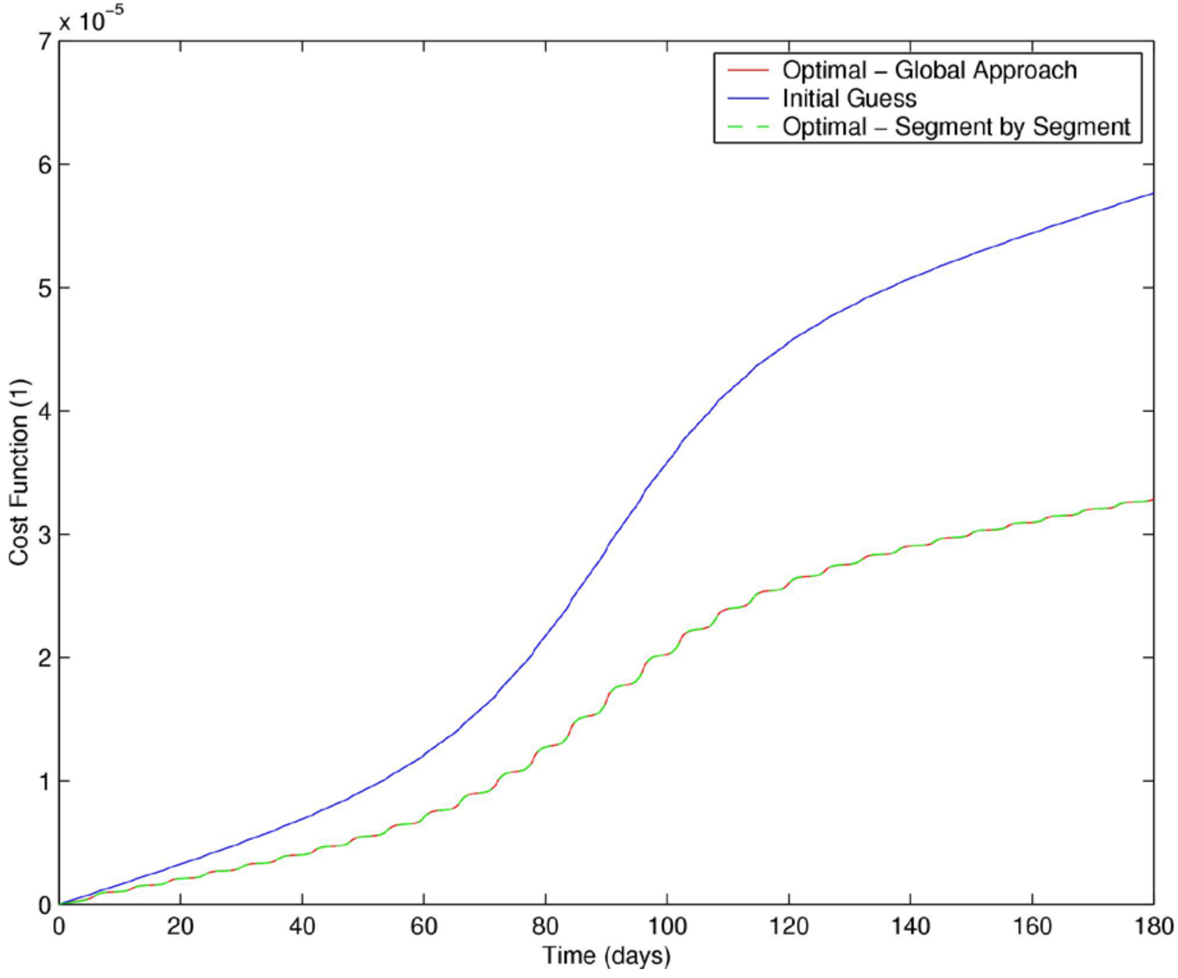
(e) J and $\dfrac{\partial H_i}{\partial \overline{u}_i}$ used in next update of control input. (Subspace Trust Region Method)

State Corrector vs. Nonlinear Optimal Control: Magnitude of Radial Error

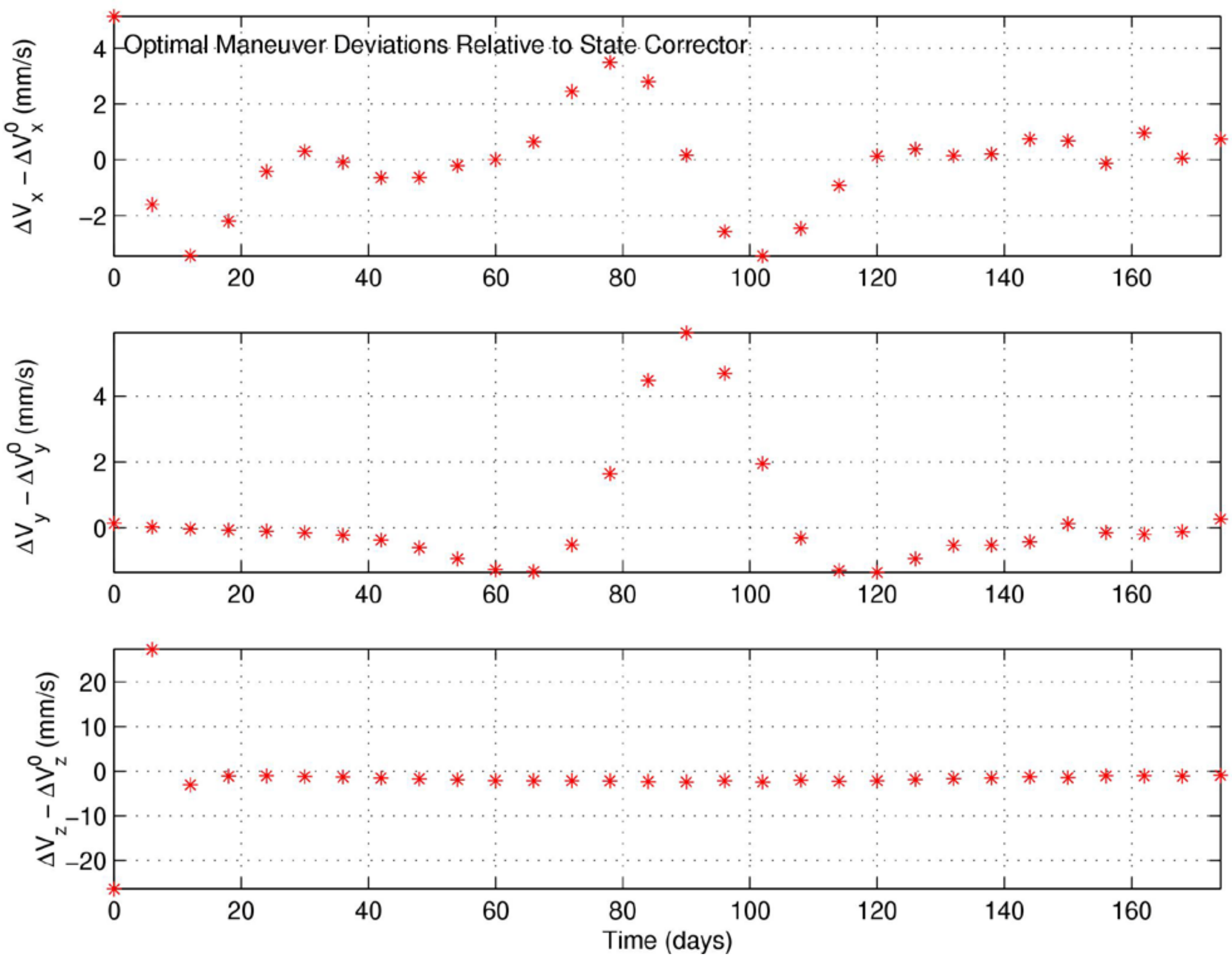
$$\min J = \sum_{i=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{1}{2}\left(\overline{x} - \overline{x}^{\circ}\right)^T Q \left(\overline{x} - \overline{x}^{\circ}\right) d$$



State Corrector vs. Nonlinear Optimal Control: Cost Function



State Corrector vs. Nonlinear Optimal Control: Impulsive Maneuver Differences



Method #2: Nonlinear Programming

- General Nonlinear Programming (NLP) Problem

$$J = \min\left(F\left(\bar{x}\right)\right);\ \ \bar{c}_L \le \bar{c}\left(\bar{x}\right) \le \bar{c}_U \quad \text{and} \ \ \bar{x}_L \le \bar{x} \le \bar{x}_U$$

  – Sequential Programming Solution → Algebraic System
    - Approximate Lagrangian to 2nd Order

$$L\left(\bar{x}\right) = F\left(\bar{x}\right) - \bar{\lambda}^T \bar{c}\left(\bar{x}\right)$$

$$L\left(\bar{x}\right) \approx \underbrace{\left.\frac{\partial L}{\partial \bar{x}}\right|^{*}}_{\text{Jacobian Matrix}} \underbrace{\left(\bar{x} - \bar{x}^{*}\right)}_{\text{Search Direction}} + \frac{1}{2}\left(\bar{x} - \bar{x}^{*}\right)^T \underbrace{\left.\left[\frac{\partial}{\partial \bar{x}}\left(\frac{\partial L}{\partial \bar{x}}\right)\right]\right|^{*}}_{\text{Hessian Matrix}} \left(\bar{x} - \bar{x}^{*}\right)$$

Jacobian Matrix

Search Direction

Hessian Matrix

    - Approximate constraints as linear
    - Iterative solution via globalized Newton methods

Dynamic Optimization via Nonlinear Programming

- Divide trajectory into phases (segments)
- Define objective function
- For each phase, define
  - Dynamic variables $\bar{z}^{(k)} = [\bar{y}^{(k)}, \bar{u}^{(k)}]$
  - State equation $\dot{\bar{y}}^{(k)} = \bar{f}^{(k)}[\bar{y}^{(k)}(t), \bar{u}^{(k)}(t), \bar{p}^{(k)}, t]$
  - Nonlinear constraints $\bar{g}_l^{(k)} \le \bar{g}^{(k)}[\bar{y}^{(k)}(t), \bar{u}^{(k)}(t), \bar{p}^{(k)}, t] \le \bar{g}_u^{(k)}$
  - State Vector Limits $\bar{y}_l^{(k)} \le \bar{y}^{(k)}(t) \le \bar{y}_u^{(k)}$
  - Control Vector Limits $\bar{u}_l^{(k)} \le \bar{u}^{(k)}(t) \le \bar{u}_u^{(k)}$
  - Phase boundary conditions $\bar{\Psi}_l \le \bar{\Psi} \le \bar{\Psi}_u$
- Approximate State Equations by Direct Transcription
- Use SOCS SQP algorithm to solve

Direct Transcription

Example: Hermite-Simpson Discretization

Given an initial guess for $\bar{y}^{(k)}$, $\bar{u}^{(k)}$ at each node, the defect ($\bar{\varsigma}_k$) at k^{th} node:

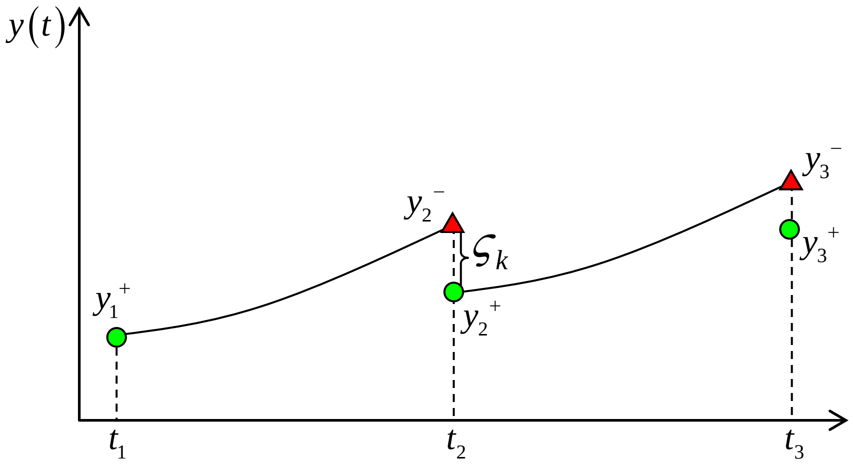
$$\bar{\varsigma}_k = \bar{y}_{k+1} - \bar{y}_k - \frac{h_k}{6}\left(\bar{f}_{k+1} + 4\bar{f}'_{k+1} + \bar{f}_k\right)$$

where

$$\bar{f}'_{k+1} = \bar{f}\left[\bar{y}'_{k+1}, \bar{u}'_{k+1}, t_k + \frac{h_k}{2}\right]$$

$$\bar{y}'_{k+1} = \frac{1}{2}\left(\bar{y}_{k+1} + \bar{y}_k\right) + \frac{h_k}{8}\left(\bar{f}_k - \bar{f}_{k+1}\right)$$

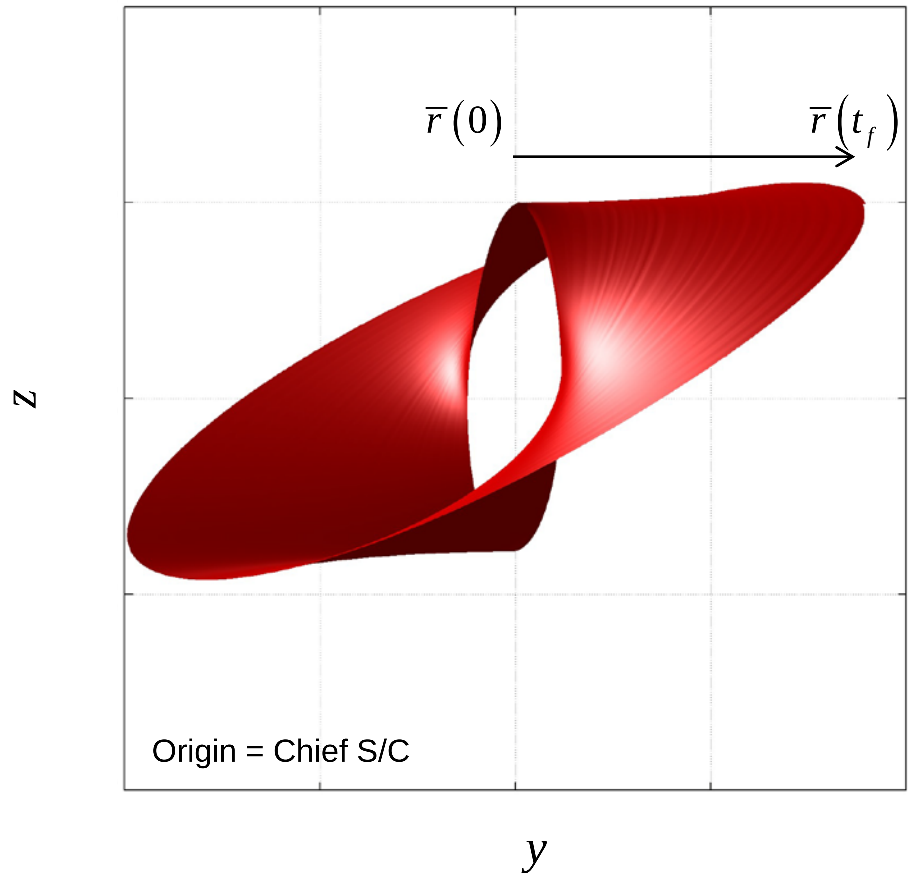
$$h_k = \tau_k \Delta t$$



- Treat the defects as a constraint, $\varsigma_k = 0$, imposed on the cost function!
- The partials of the defect equations lead to large sparse matrices.
- Use SOCS (Sparse Optimal Control Software) to ID solution.

Sample Startup Solution: Slowly Drifting Vertical Orbit

100 Revolutions = 18,000 days



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Example 2: Continuous Optimal Control Goal → Periodicity

$$J = \sum_{k=1}^{N} \int_{t_i^{(k)}}^{t_f^{(k)}} \bar{u}^T(t)\bar{u}(t)\,dt$$

Constraints:

$$t_i^{(k)} = \text{fixed}, \quad \text{for } k = 1,\ldots,4$$

$$t_f^{(k)} = \text{fixed}, \quad \text{for } k = 1,\ldots,4$$

$$0 = \bar{r}\left(t_i^{(1)}\right) - \bar{r}_i$$

$$0 = \bar{u}\left(t_i^{(1)}\right)$$

$$0 = \bar{z}\left(t_f^{(k)}\right) - \bar{z}\left(t_i^{(k+1)}\right), \quad k = 1,\ldots,3$$

$$0 = \bar{y}\left(t_i^{(1)}\right) - \bar{y}\left(t_f^{(4)}\right)$$

Search Space:

$$\begin{bmatrix} -10{,}000 \text{ km} \\ -10{,}000 \text{ km} \\ -10{,}000 \text{ km} \\ -4 \text{ m/sec} \\ -4 \text{ m/sec} \\ -4 \text{ m/sec} \end{bmatrix} \le \bar{y}(t) \le \begin{bmatrix} 10{,}000 \text{ km} \\ 10{,}000 \text{ km} \\ 10{,}000 \text{ km} \\ 4 \text{ m/sec} \\ 4 \text{ m/sec} \\ 4 \text{ m/sec} \end{bmatrix}$$

$$\begin{bmatrix} -4 \text{ m/s}^2 \\ -4 \text{ m/s}^2 \\ -4 \text{ m/s}^2 \end{bmatrix} \le \bar{u}(t) \le \begin{bmatrix} 4 \text{ m/s}^2 \\ 4 \text{ m/s}^2 \\ 4 \text{ m/s}^2 \end{bmatrix}$$

Numerical Issues

- Relative Scaling Problems
  - Non-convergence
  - All constraints met except for control acceleration continuity
- Source
  - Small control accelerations trick the software into convergence
- Solution
  - Chief S/C path pre-determined and stored using B-splines
  - Internal rescaling of variables
  - Use dimensional form of relative equations of motion

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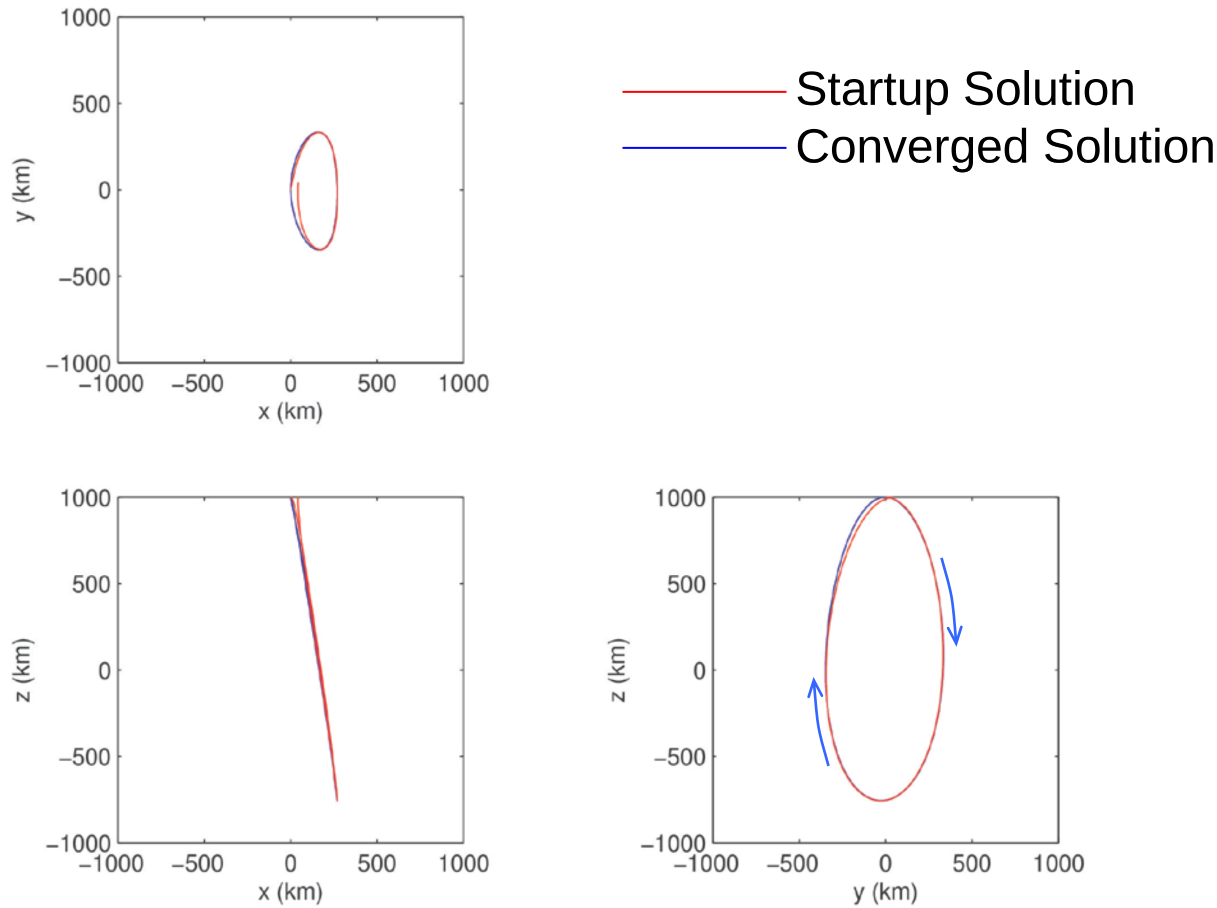
EX2: Periodicity Via Continuous Control



Conclusions

- Direct Transcription Method
  - With proper variable scaling, responds well to dynamical sensitivity of n-body problem.
  - Accuracy issues overcome through mesh refinement.
  - Availability of higher order representations may be useful in reducing mesh refinement iterations. These methods not currently present in SOCS.
- Partial Discretization Method
  - Similar optimization scheme in some respects
  - No constraints presently included in the formulation
  - Solution speed hindered by sequencing
  - Accuracy controlled by integrator selection

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Backups

Distributed S/C Systems

- Generic formulation → Application Independent
  - Formation Flight
    - Vehicles share and exchange information to accomplish mission
      - Central vehicle → Chief S/C
      - Other vehicles → Deputies
      - Examples
        - » Interferometry
        - » Surface Imaging
        - » Radar
        - » Geolocation
  - Vehicle Rendezvous & Docking
    - Resources and information may be transferred (application dependent)
      - Central "chief" vehicle → not necessarily aware of the presence or activities of other spacecraft or "deputies".
      - Deputy vehicles → perform operations on or in the vicinity of the chief
      - Examples:
        - » Resource transfer (fuel, equipment, etc.) between vehicles
        - » Unmanned on-orbit servicing of satellites
        - » Space based threat assessment and handling

Relative Dynamics

(Frame Independent Formulation)

Define the mathematical model that preserves generality for all apps/systems.

Absolute Dynamical Model

Nonlinear System

$$\dot{\bar{y}}_c = \bar{f}\left(\bar{y}_c\right)$$

$$\dot{\bar{y}}_d = \bar{f}\left(\bar{y}_d\right) + B\bar{u}\left(t\right)$$

Linear System

$$\delta\dot{\bar{y}}_c\left(t\right) = A_c\left(t\right)\delta\bar{y}_c\left(t\right)$$

$$\delta\dot{\bar{y}}_d\left(t\right) = A_d\left(t\right)\delta\bar{y}_d\left(t\right) + B\delta\bar{u}\left(t\right)$$

Relative Dynamical Model

Nonlinear System

$$\dot{\bar{x}} = \bar{f}\left(\bar{y}_d\right) - \bar{f}\left(\bar{y}_c\right) + \bar{u}\left(t\right)$$

$$\dot{\bar{x}} = \bar{f}\left(\bar{y}_c + \bar{x}\right) - \bar{f}\left(\bar{y}_c\right) + \bar{u}\left(t\right)$$

$$\dot{\bar{x}} = \bar{F}\left(\bar{x}, \bar{u}, \bar{y}_c\right)$$

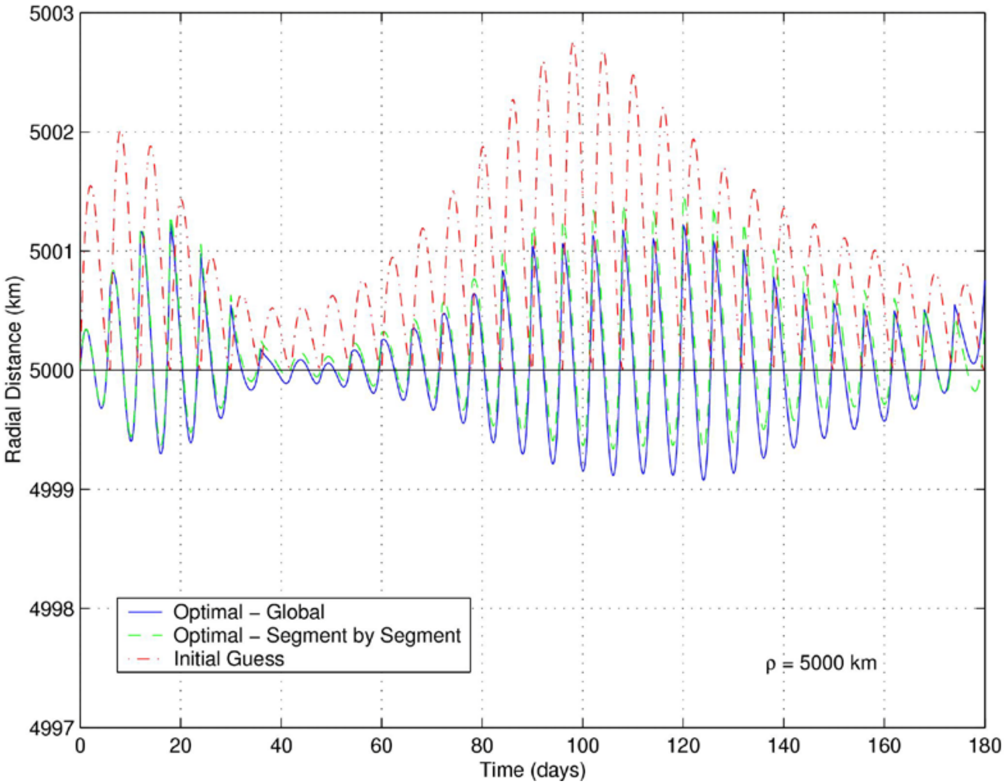
Linear System

$$\delta\dot{\bar{x}}\left(t\right) = \frac{\partial\bar{F}}{\partial\bar{x}}\delta\bar{x}\left(t\right) + \frac{\partial\bar{F}}{\partial\bar{u}}\delta\bar{u}\left(t\right)$$

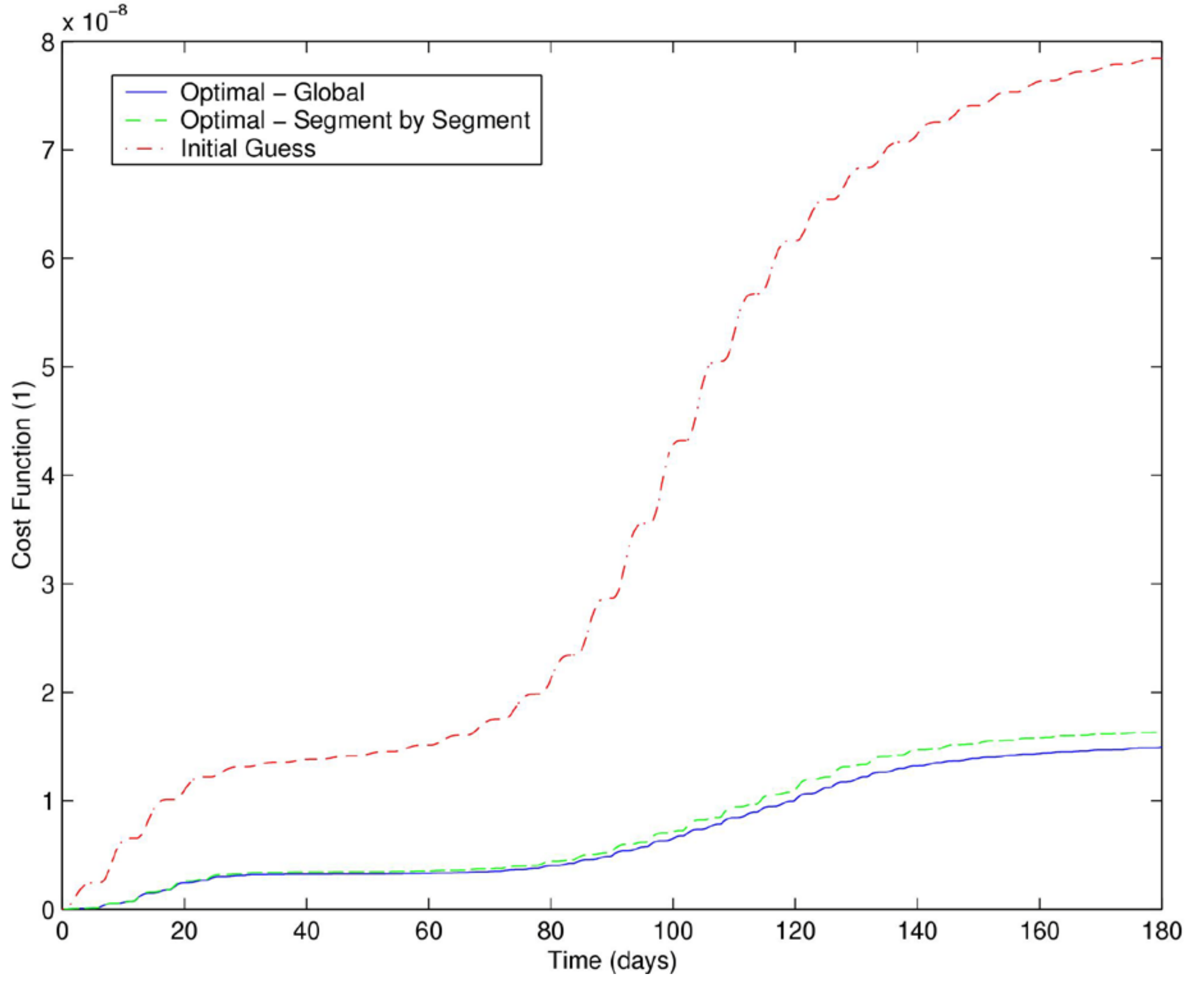
$$= A_d\left(t\right)\delta\bar{x}\left(t\right) + B\delta\bar{u}\left(t\right)$$

Impulsive Radial Optimal Control

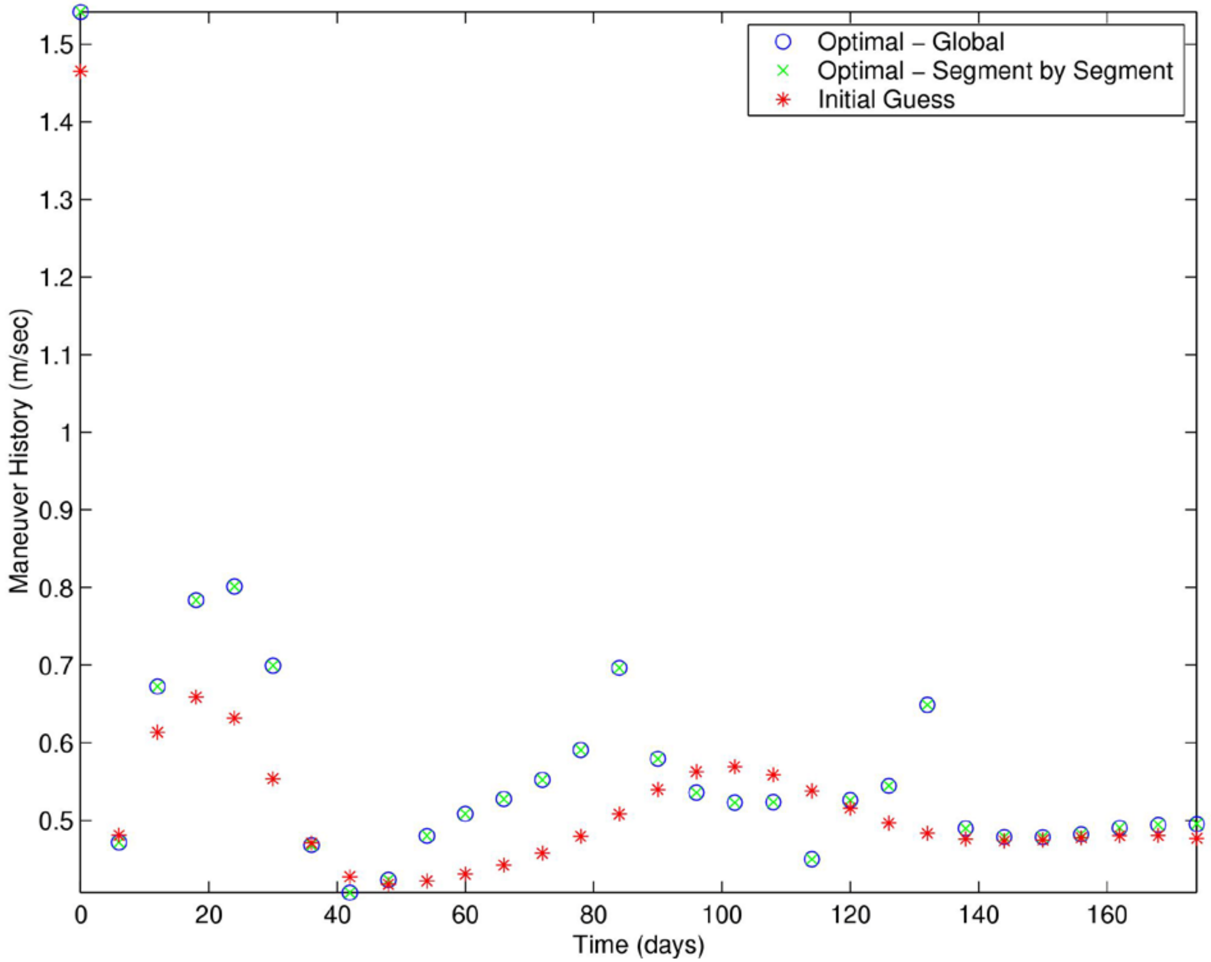
$$\min J = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{1}{2} q \left(r - r^{\circ} \right)^2 dt$$



Radial Optimal Control: Cost Functional



Radial Optimal Control: Maneuver History



Dynamic Optimization Approach

Direct

Find a sequence of points $z_1, z_2, \ldots, z^*$ such that $F(z_1) > F(z_2) > \ldots > F(z^*)$.

This only requires a comparison of the objective function at each point.

Indirect

Identify the root of the necessary condition $F'(z) = 0$. This requires that the user compute the derivative of the cost index and determine if it meets the specified tolerance.

From an optimal control perspective:

Indirect Optimization $\rightarrow$ Identify the roots of the KKT Conditions (Euler-Lagrange)

Direct Optimization $\rightarrow$ Does NOT require the explicit derivation and construction of the necessary conditions (i.e. the adjoint equations, the control equations, or the transversality conditions) that are required by the Euler-Lagrange equations.

Direct vs. Indirect

- If the optimality conditions are already determined by the EL-equations, why not use an indirect method? For the formation keeping problem, this approach has been investigated but is not recommended.
  - The partial derivative matrices, in this case, involve a matrix quadrature of a function of the STM. This is computationally intensive of course.
  - Requires a good estimate of the constrained trajectory arc to start the optimization process.
  - In general, the numerical process is extremely sensitive (ill-conditioned) to the initial guess for the Lagrange multipliers. This problem is even more difficult to deal with in the n-body problem.
- Since, in the n-body problem, an exact solution is not available for the KKT equations, a direct method is better suited for nonlinear optimization in this case.

Direct Methods: Nonlinear Programming

- Parameter Optimization
  - Finite dimensional
  - Solution → Globalized Newton Methods
- Functional Optimization → Optimal Control
  - Infinite dimensional
  - Solution → Transcribe into finite dimensional problem
    - Represent dynamical system in terms of finite set of variables
    - Solve the finite dimensional problem using NLP
      - Reduces problem to solving an algebraic system of eqns.
    - Assess accuracy of finite dimensional approximation
    - If needed, refine grid and repeat first two steps

Optimal Control Preliminaries

- Observations
  - The cost index depends on point functions and quadrature functions
  - Nonlinear point functions can include variables from all phases
  - Quadrature functions are evaluated along the length of the phase by augmenting the state vector:

$$F\left[\bar{y}^{(k)}(t),\bar{u}^{(k)}(t),\bar{p}^{(k)},t\right]=\begin{bmatrix}\bar{f}\left[\bar{y}^{(k)}(t),\bar{u}^{(k)}(t),\bar{p}^{(k)},t\right]\\ w\left[\bar{y}^{(k)}(t),\bar{u}^{(k)}(t),\bar{p}^{(k)},t\right]\end{bmatrix}$$

  - The boundary conditions also depend on variables from all phases
  - Each phase is divided into *N* mesh points for the discretization
  - Each interior grid point is assigned a control variable, $\bar{u}^{(k)}(t_j)=\bar{u}_j^{(k)}$

Define Optimal Control Problem

For each phase, k, define a vector of dynamic variables, $\bar{z}^{(k)}(t) = \left[\bar{y}^{(k)}(t), \bar{u}^{(k)}(t)\right]$

that includes both the state vector, $\bar{y}^{(k)}(t)$, and the control input vector, $\bar{u}^{(k)}(t)$.

$$\min(J) = \Phi\left[\bar{z}_i^{(1)}, t_i^{(1)}, \bar{z}_f^{(1)}, t_f^{(1)}, \bar{p}^{(1)}, \ldots, \bar{z}_i^{(N)}, t_i^{(N)}, \bar{z}_f^{(N)}, t_f^{(N)}, \bar{p}^{(N)}\right] + \sum_{j=1}^{N} \int_{t_I^{(j)}}^{t_F^{(j)}} w^{(j)}\left[\bar{z}^{(j)}(t), t^{(j)}, \bar{p}^{(j)}\right] dt$$

Each phase is subject to:

$$\dot{\bar{y}}^{(k)}(t) = \bar{f}\left(\bar{z}^{(k)}(t), \bar{p}^{(k)}, t\right); \quad t_i^{(k)} \le t \le t_f^{(k)}$$

$$\bar{g}_l^{(k)} \le \bar{g}^{(k)}\left(\bar{z}^{(k)}(t), \bar{p}^{(k)}, t\right) \le \bar{g}_u^{(k)}$$

$$\bar{z}_l^{(k)} \le \bar{z}^{(k)}(t) \le \bar{z}_u^{(k)}$$

The phases are linked by boundary conditions of the form:

$$\bar{\Psi}_l \le \bar{\Psi}\left[\bar{z}_i^{(1)}, t_i^{(1)}, \bar{z}_f^{(1)}, t_f^{(1)}, \bar{p}^{(1)}, \ldots, \bar{z}_i^{(N)}, t_i^{(N)}, \bar{z}_f^{(N)}, t_f^{(N)}, \bar{p}^{(N)}\right] \le \Psi_u$$

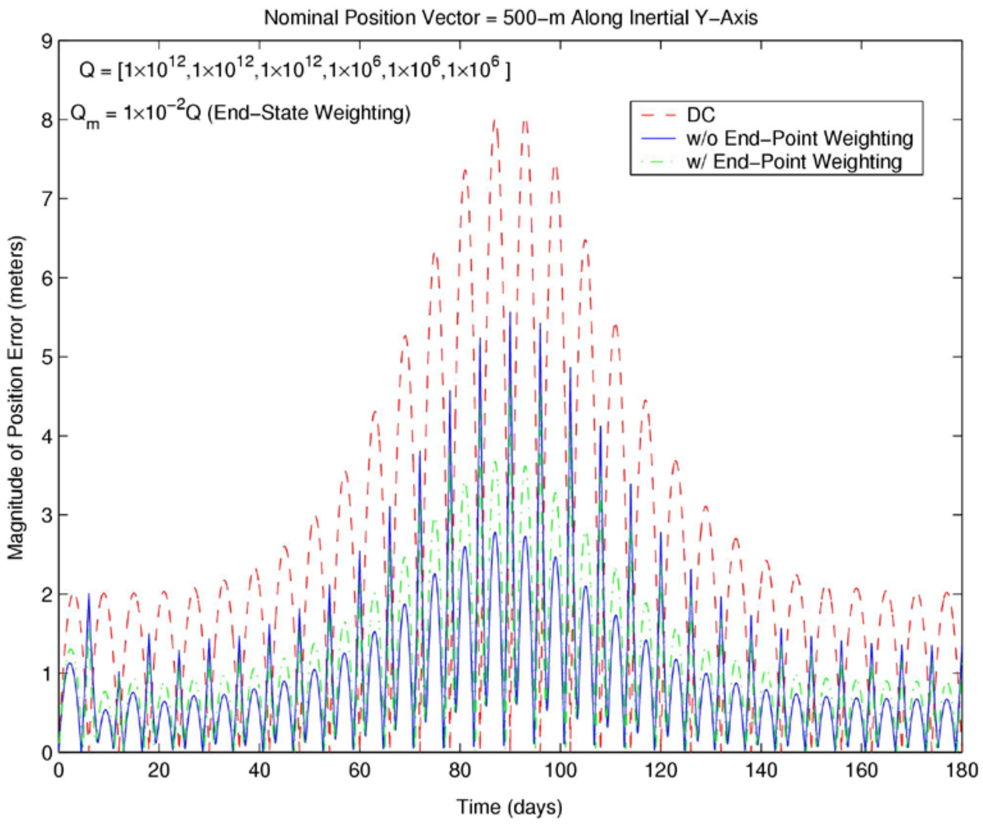
Optimizer Test

- Identification of a good startup solution for the optimizer is necessary to ensure convergence.
  - Determine non-periodic but bounded relative orbits in the linearized system using the Floquet Controller.
  - Employ a 2-level differential corrections process to converge the solution in the nonlinear system.
  - Transfer this solution as an initial guess to the nonlinear optimal control process.
  - Choose mathematical model that is consistent with ephemeris formulation for later transition into the Generator FORMATION tool.
  - Impose closed-path constraint as a test case.

Impulsive Optimal Control

Minimize State Error with End-State Weighting

$$\min J = \frac{1}{2}\left(\bar{x}_N - \bar{x}_N^\circ\right)^T W\left(\bar{x}_N - \bar{x}_N^\circ\right) + \sum_{i=0}^{N-1}\int_{t_j}^{t_{j+1}} \frac{1}{2}\left(\bar{x} - \bar{x}^\circ\right)^T Q\left(\bar{x} - \bar{x}^\circ\right) d$$

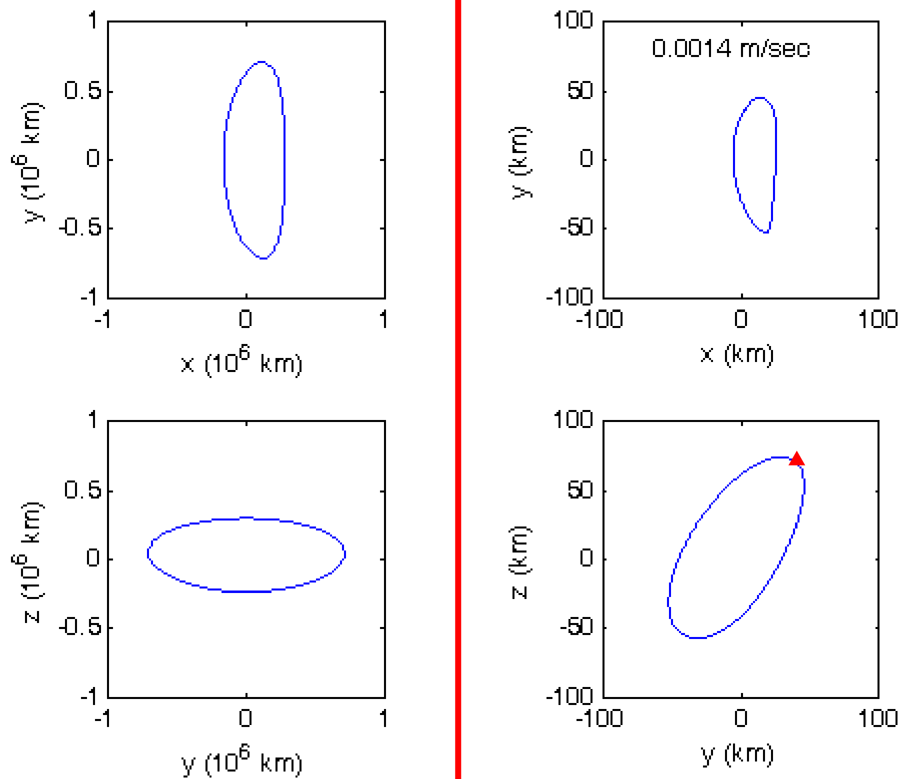


$$\bar{\rho} = (500 \text{ m})\hat{Y}$$

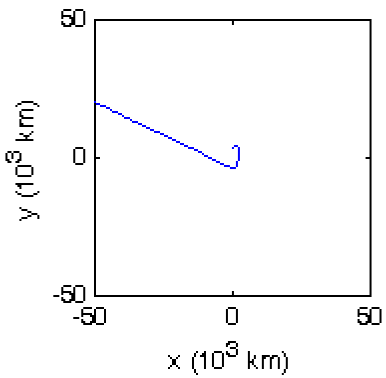
EX1: Impulsive Optimal Control: Closed Relative Path (Small)

Chief S/C Halo Orbit
A_z = 300,000 km

Deputy S/C Relative Orbit
r_{max} = 82 km

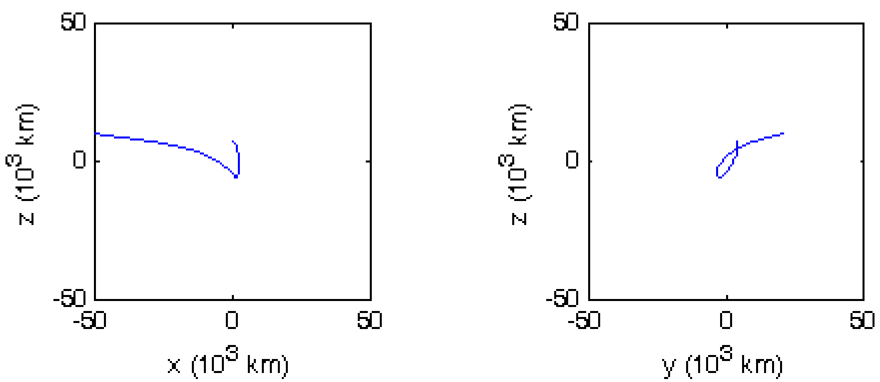


EX3: Sensitivity of Solution to Initial Guess



$$\left|\overline{r}_0\right| = 8254 \text{ km}$$

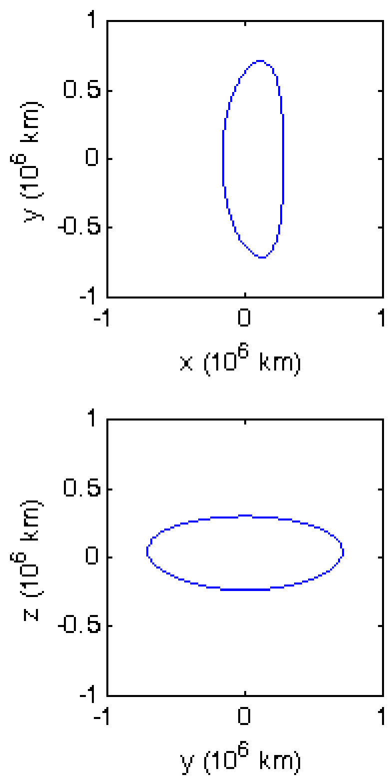
Goal: Determine min(ΔV) for $\overline{r}_0 = \overline{r}_f$



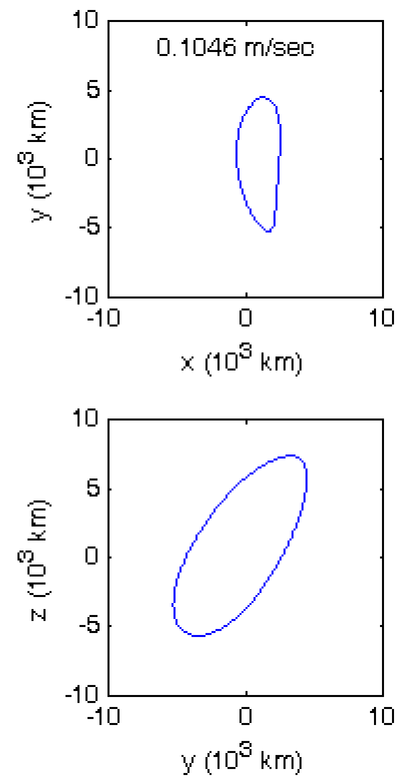
Given a bad initial guess for the optimizer …

Converged Periodic Solution (Max. Amplitude 8254 km)

Chief S/C Halo Orbit
A_z = 300,000 km



Deputy S/C Relative Orbit
r_{max} = 8254 km



… the numerical process is still able to identify the desired solution

Example #1: Impulsive Optimal Control to Achieve Closed Path

Cost Index: $\min J = \Delta\bar{V}_0^{\,T} \Delta\bar{V}_0$

Dynamical Constraint:

$$\dot{\bar{y}} = \bar{f}\left(\bar{y}, \bar{y}_c\right) = \begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{V}} \end{bmatrix}; \qquad \bar{x}(0) = \begin{bmatrix} \bar{r}_0 \\ \bar{V}_0^{\,-} \end{bmatrix}$$

Terminal Path Constraint:

$$\bar{r}_0 - \bar{r}_f = \bar{0}$$

Initial Velocity Constraint:

$$\bar{V}_0^{\,-} + \Delta\bar{V}_0 - \Delta\bar{V}_0^{\,+} = \bar{0}$$

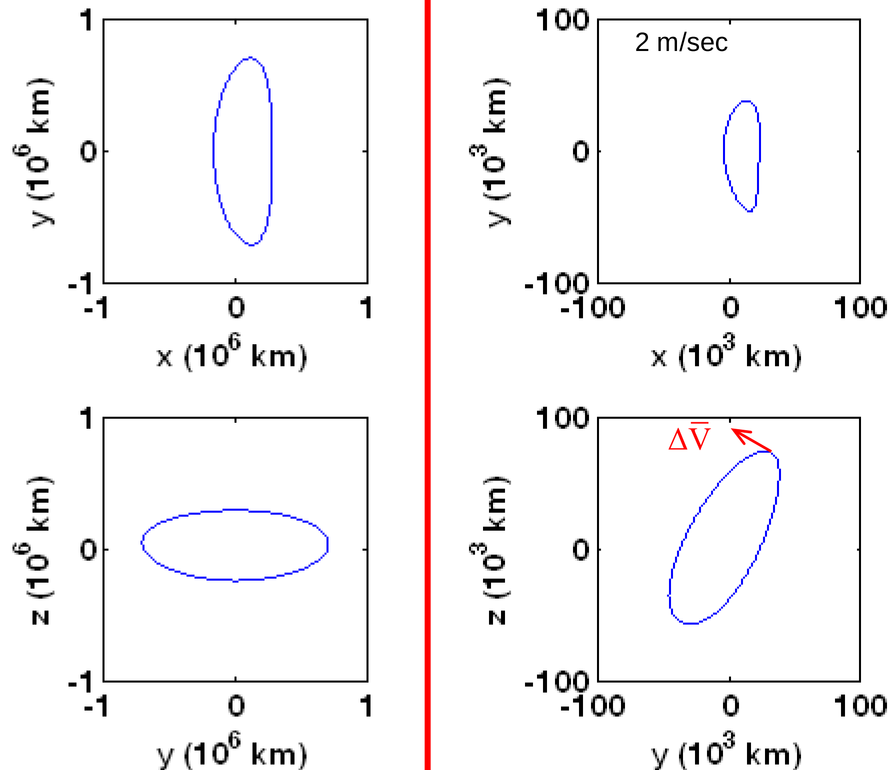
Continuity Constraints:

$$\bar{y}\left(t_f^{\,(k)}\right) - \bar{y}\left(t_i^{\,(k+1)}\right) = \bar{0}$$

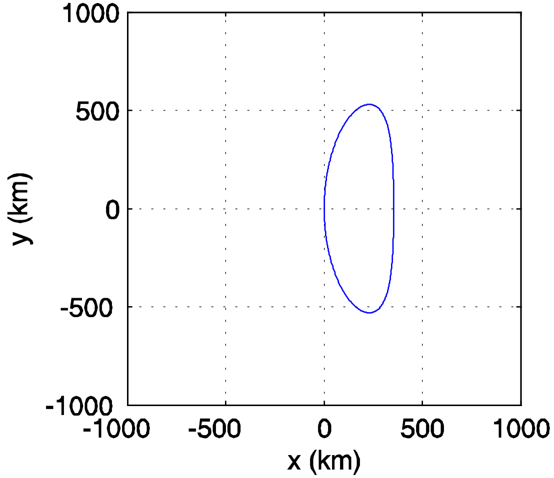
EX1: Impulsive Optimal Control: Closed Relative Path (Large)

Chief S/C Halo Orbit
A_z = 300,000 km

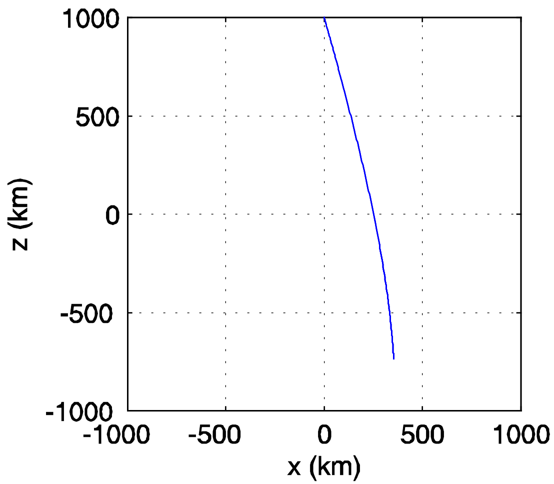
Deputy S/C Relative Orbit
r_{max} = 81057.8 km

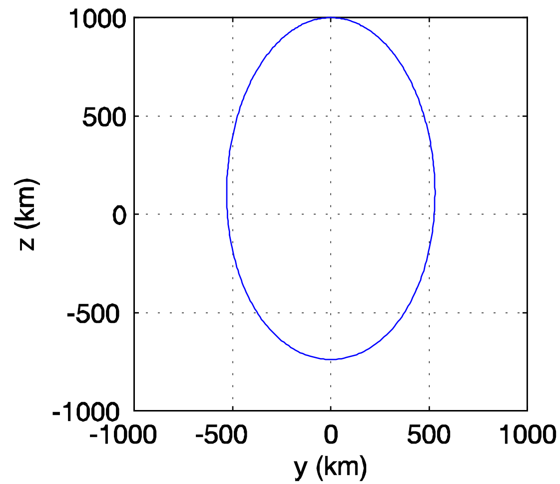


Deputy S/C Path

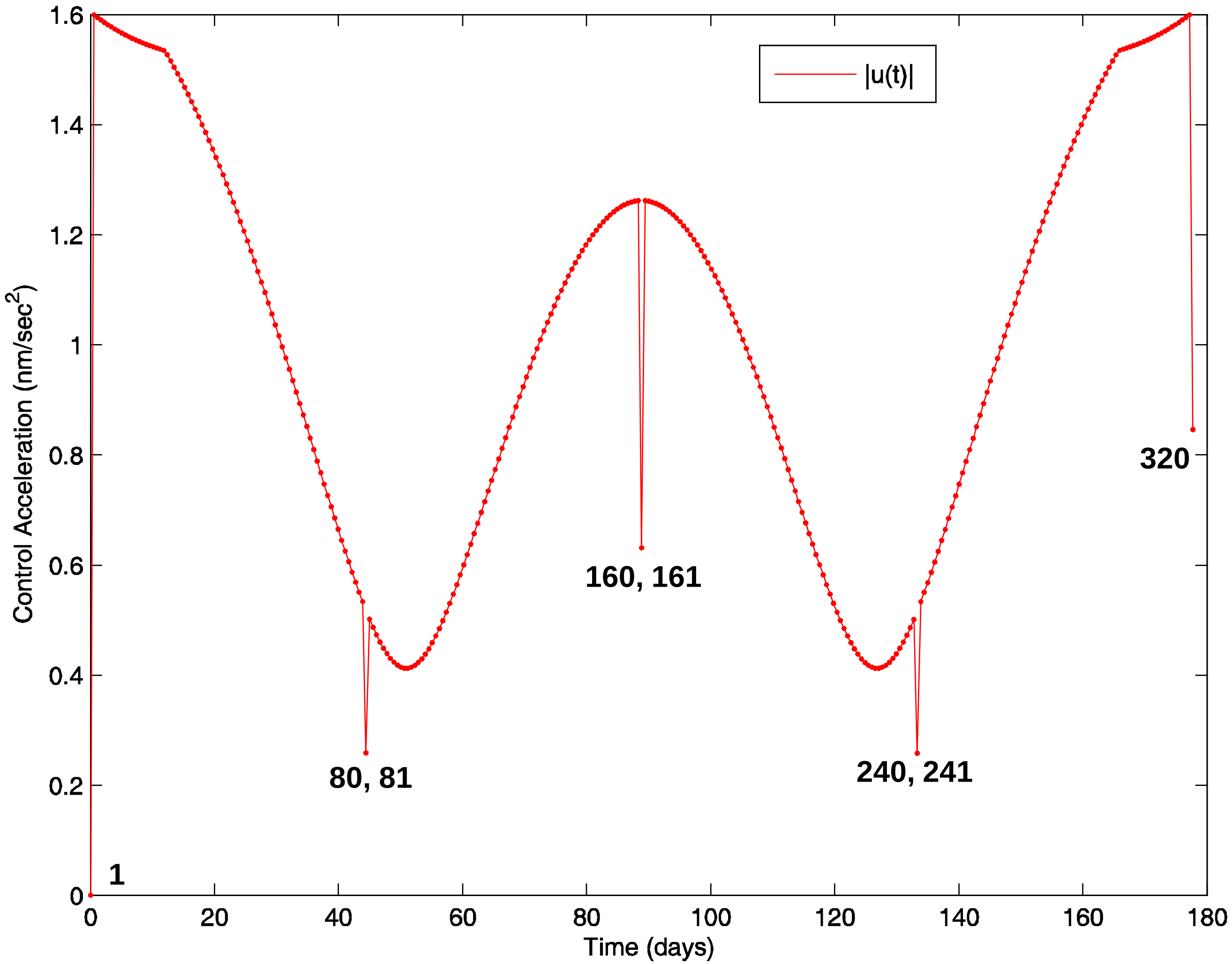


Chief S/C @ Origin





Discontinuities in Control Acceleration



EX2: Control Acceleration Profile