# DESIGN AND CONTROL OF FORMATIONS NEAR THE LIBRATION POINTS OF THE SUN-EARTH/MOON EPHEMERIS SYSTEM

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# Reference Motions

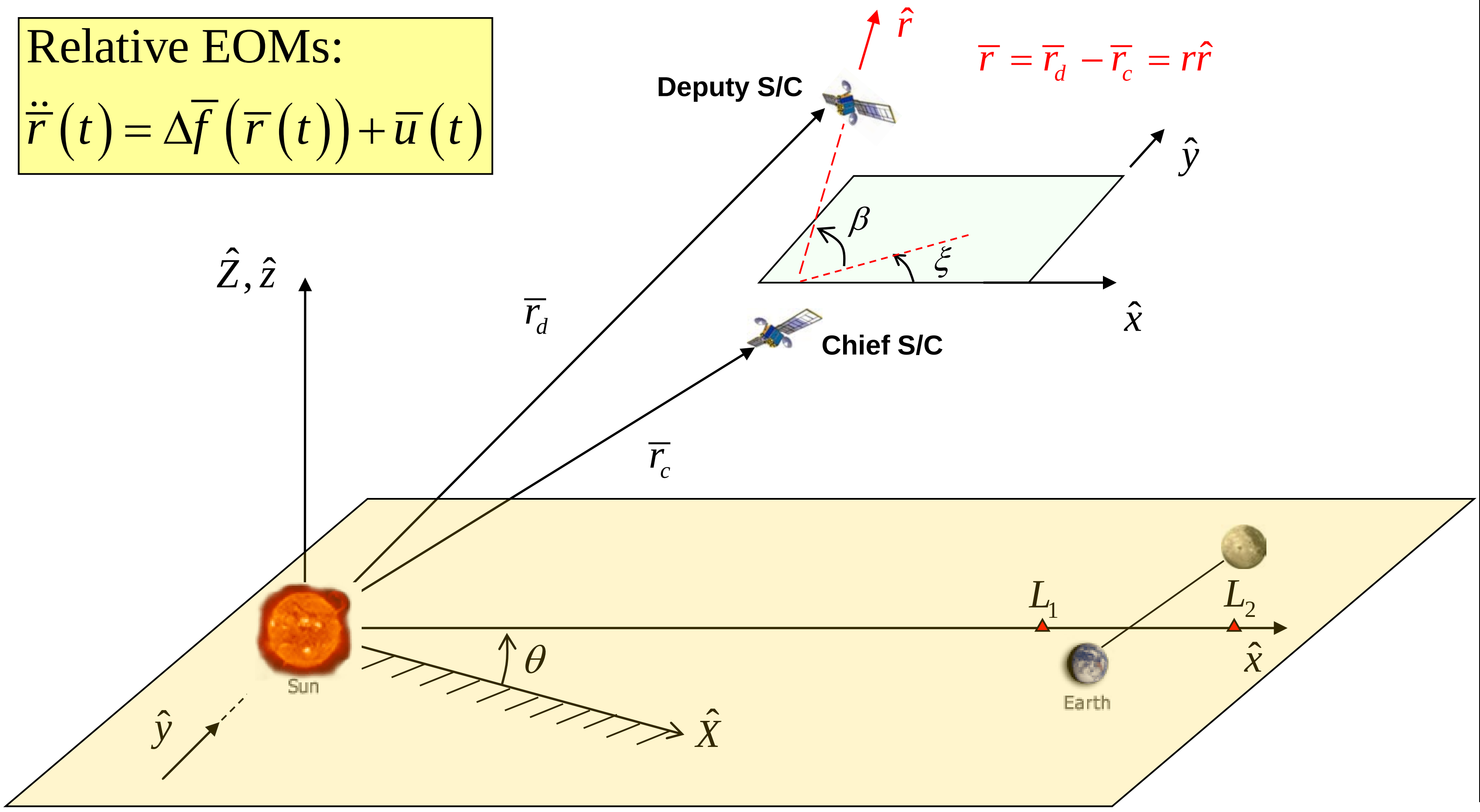
- Natural Formations
  - String of Pearls
  - Others: Identify via Floquet controller (CR3BP)
    - Quasi-Periodic Relative Orbits (2D-Torus)
    - Nearly Periodic Relative Orbits
    - Slowly Expanding Nearly Vertical Orbits

    + Stable Manifolds
- Non-Natural Formations
  - Fixed Relative Distance and Orientation { RLP / Inertial }
  - Fixed Relative Distance, Free Orientation
  - Fixed Relative Distance & Rotation Rate
  - Aspherical Configurations (Position & Rates)

# 2-S/C Formation Model in the Sun-Earth-Moon System

Relative EOMs:

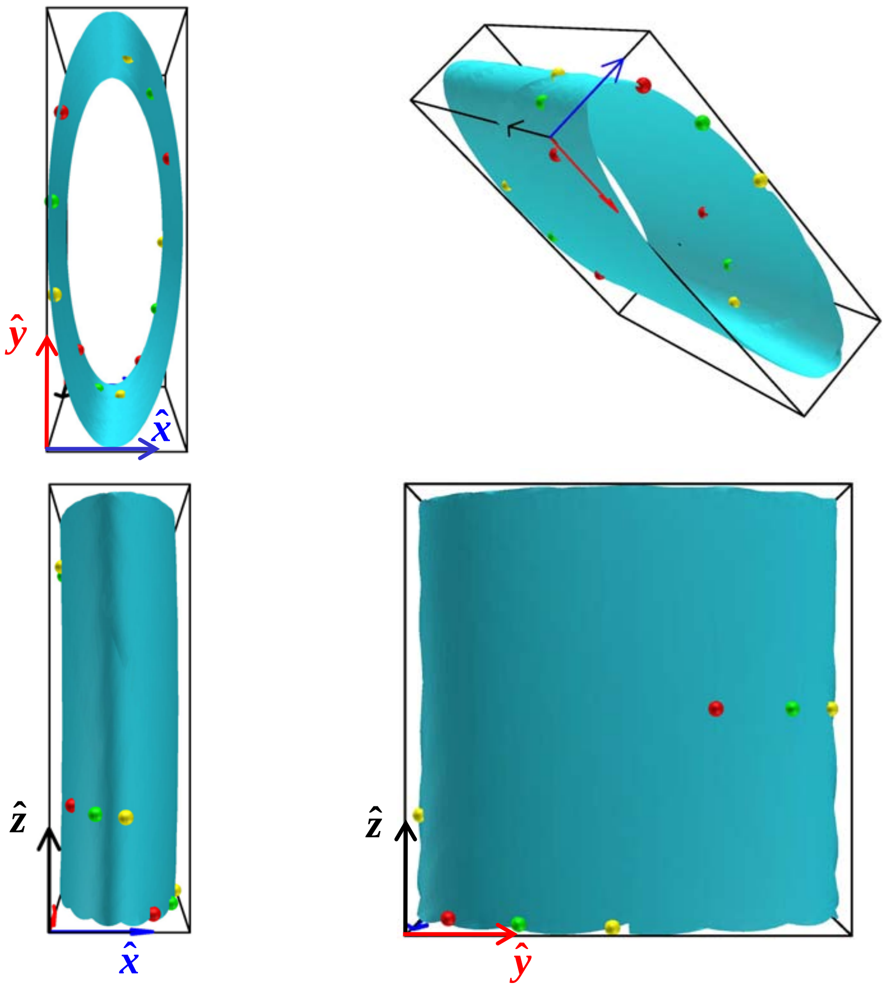
$$\ddot{\bar{r}}(t) = \Delta\bar{f}\left(\bar{r}(t)\right) + \bar{u}(t)$$




Ephemeris System = Sun+Earth+Moon Ephemeris + SRP

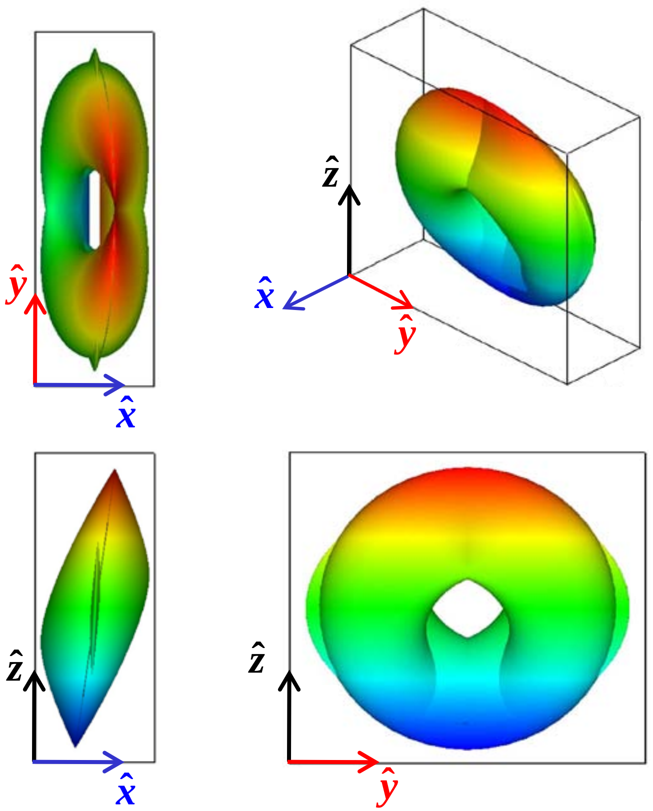
# Natural Formations

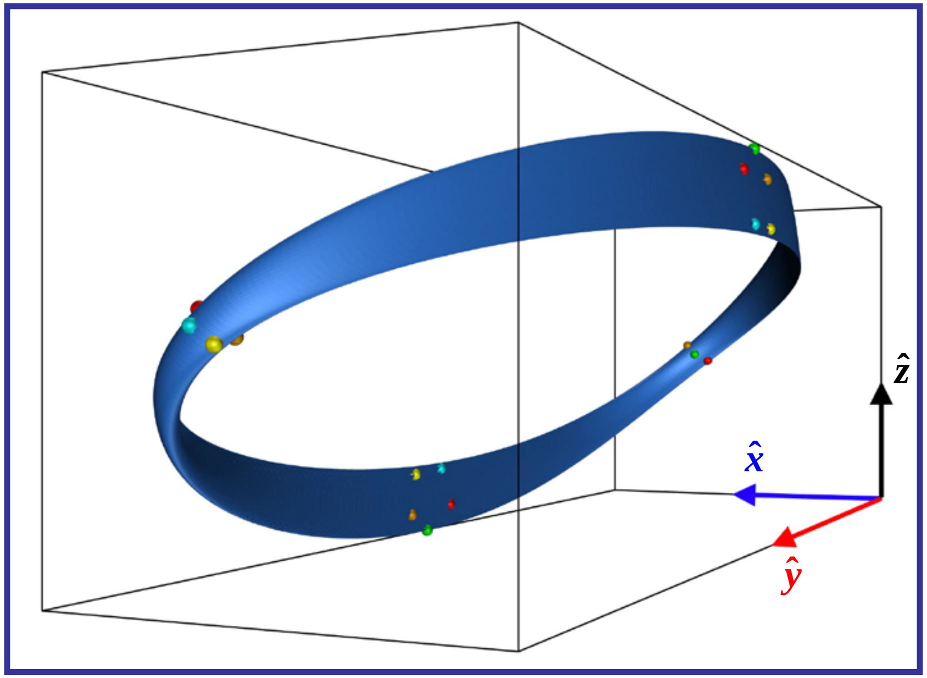
# Natural Formations: String of Pearls



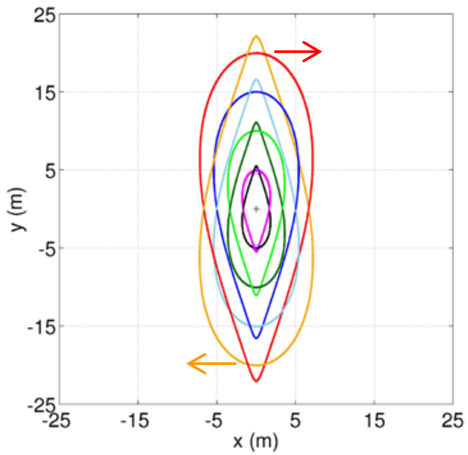
# Natural Formations: Quasi-Periodic Relative Orbits → 2-D Torus

**Chief S/C Centered View (RLP Frame)**

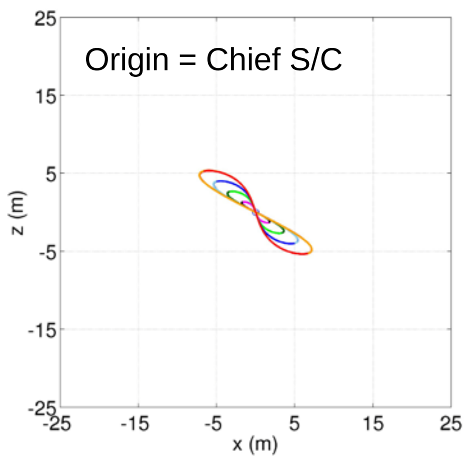


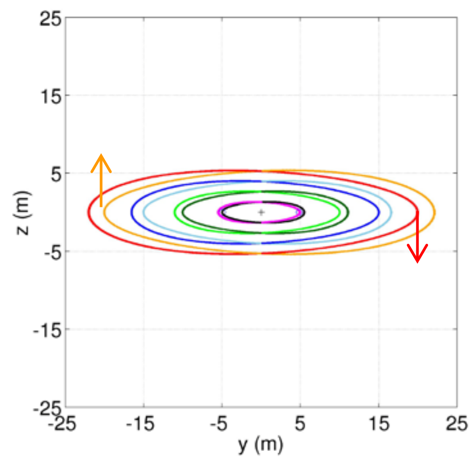



# Natural Formations:
# Nearly Periodic Relative Motion

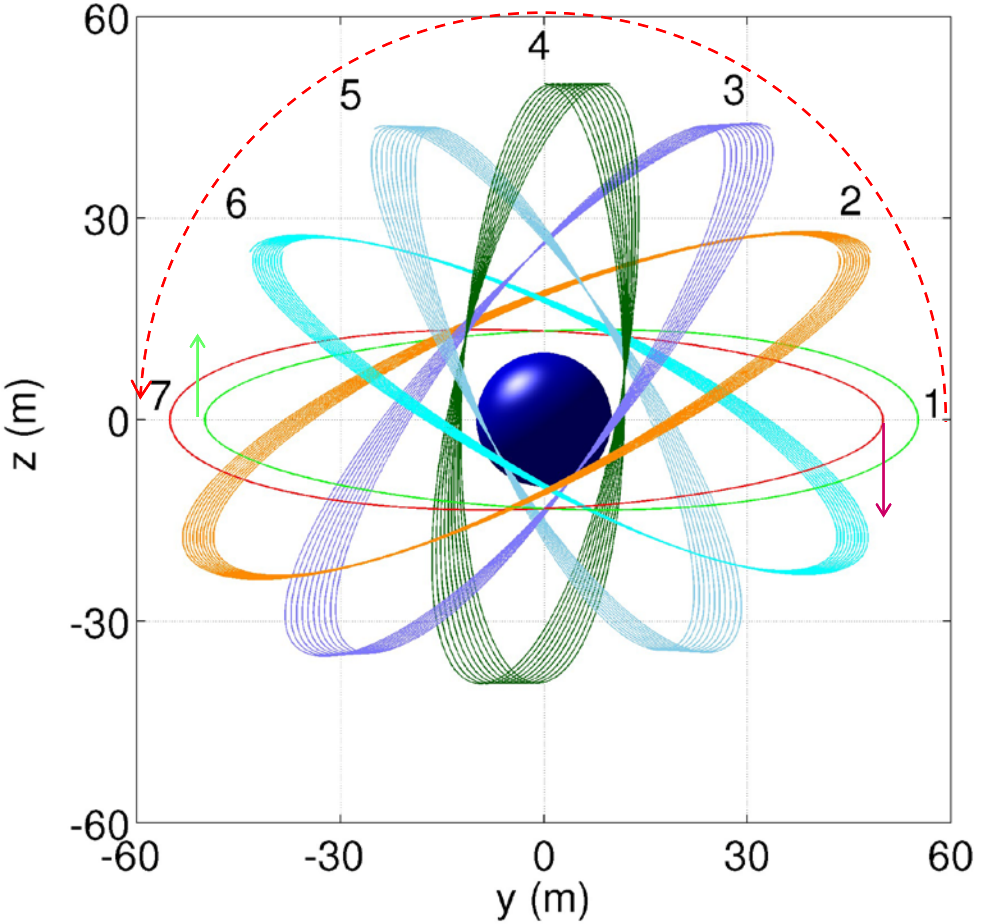



10 Revolutions = 1,800 days

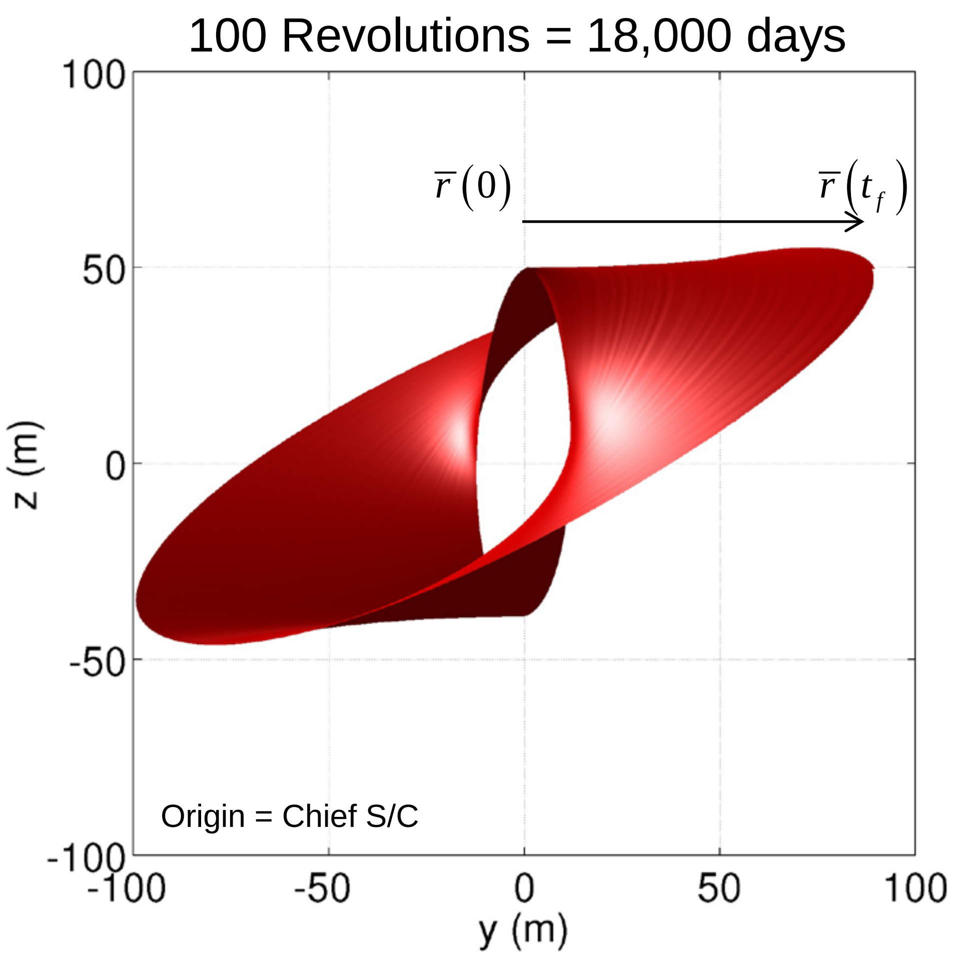


# Evolution of Nearly Vertical Orbits Along the *yz*-Plane



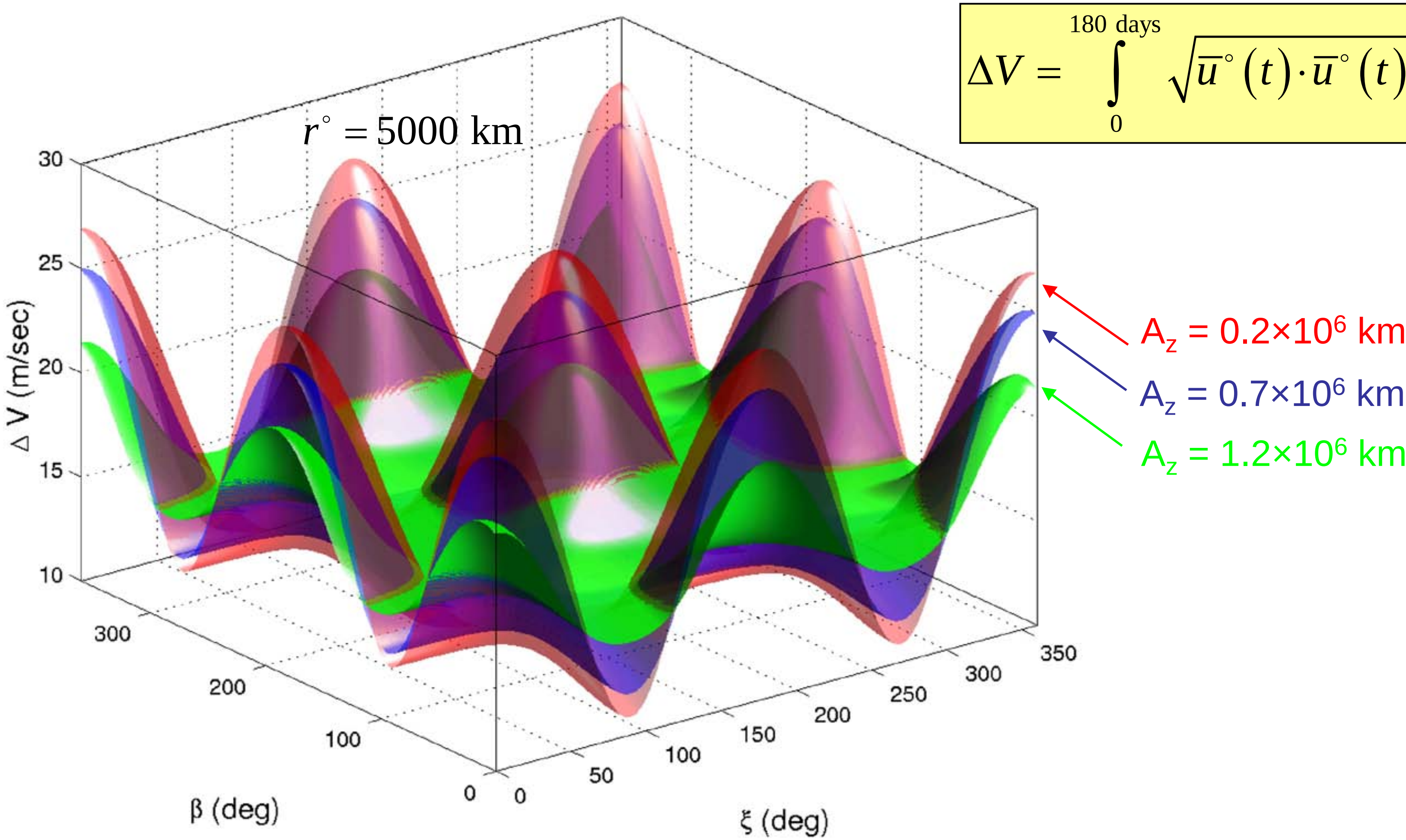
# Natural Formations:
# Slowly Expanding Vertical Orbits



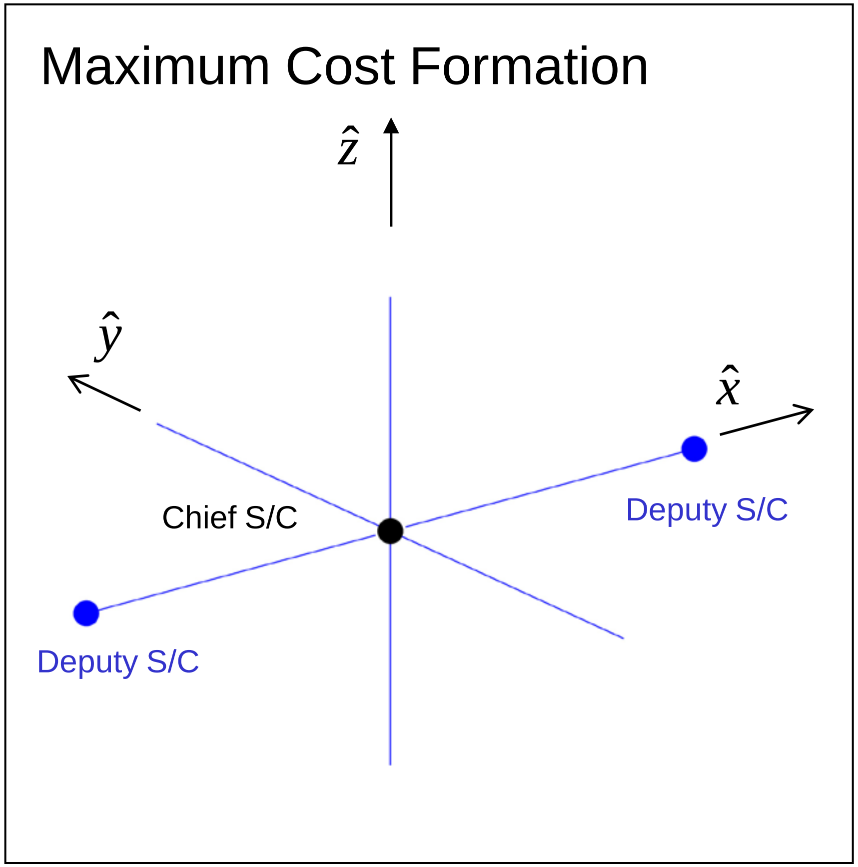
# Non-Natural Formations

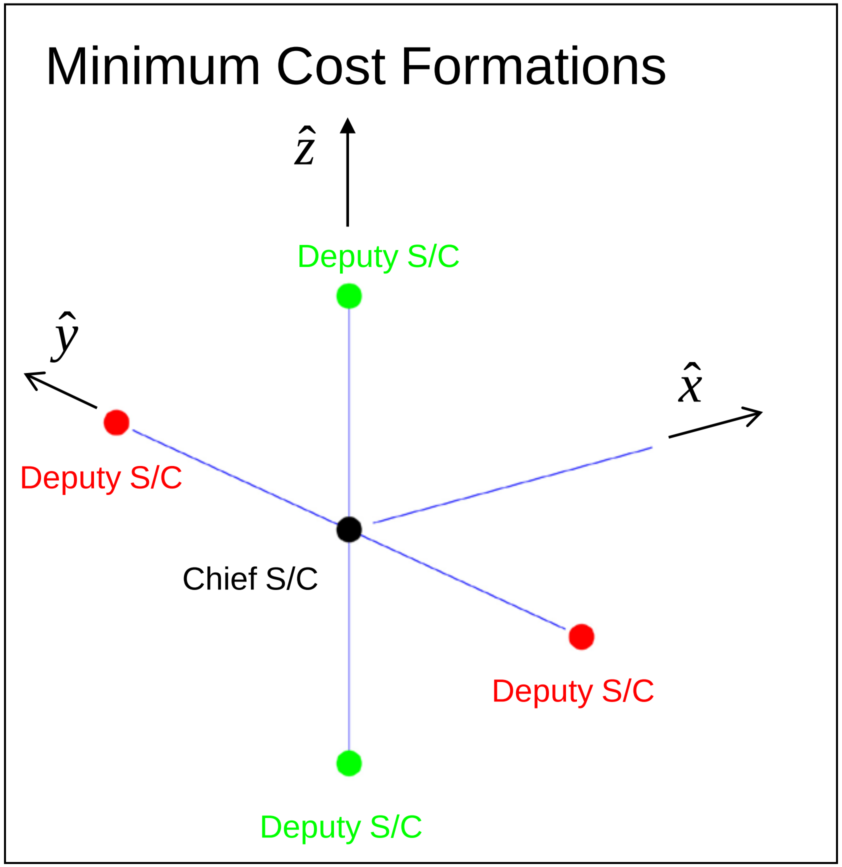
# Nominal Formation Keeping Cost
## (Configurations Fixed in the RLP Frame)

$$\Delta V = \int_{0}^{180\ \text{days}} \sqrt{\bar{u}^{\circ}(t)\cdot\bar{u}^{\circ}(t)}\, d$$



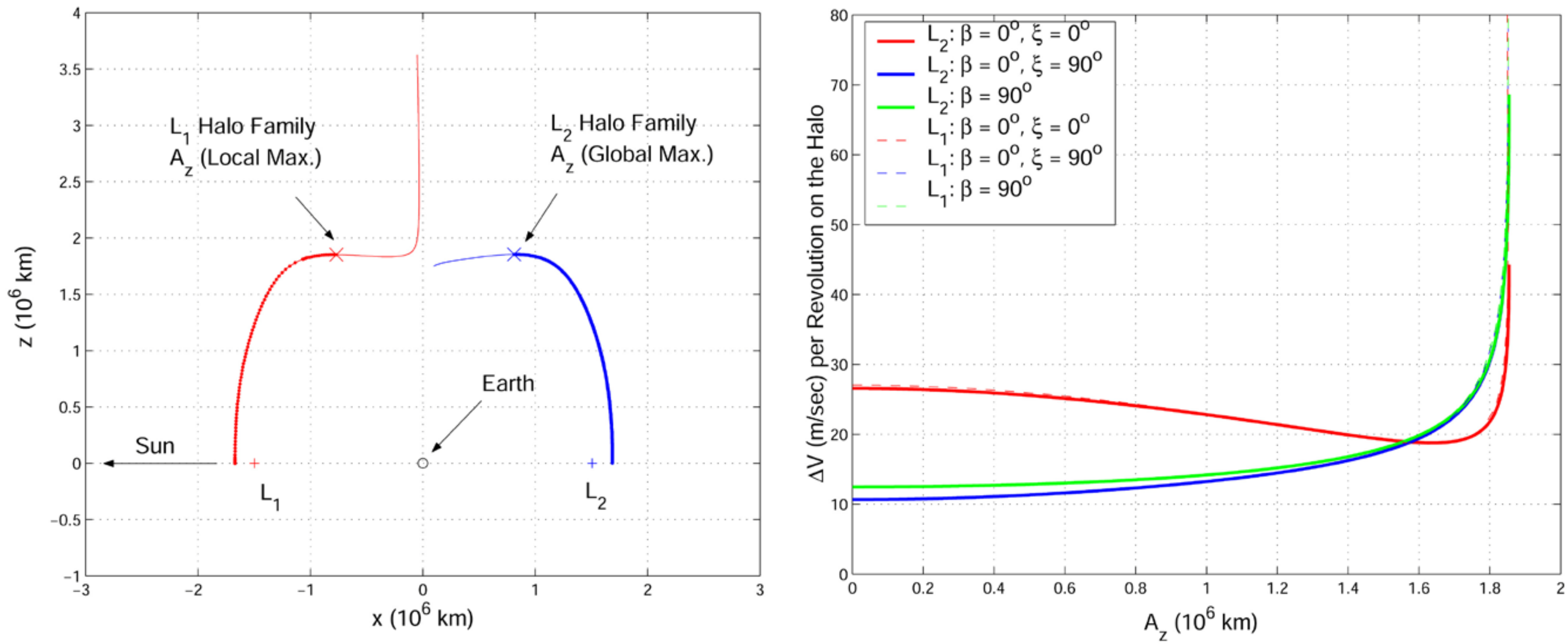
# Max./Min. Cost Formations
## (Configurations Fixed in the RLP Frame)



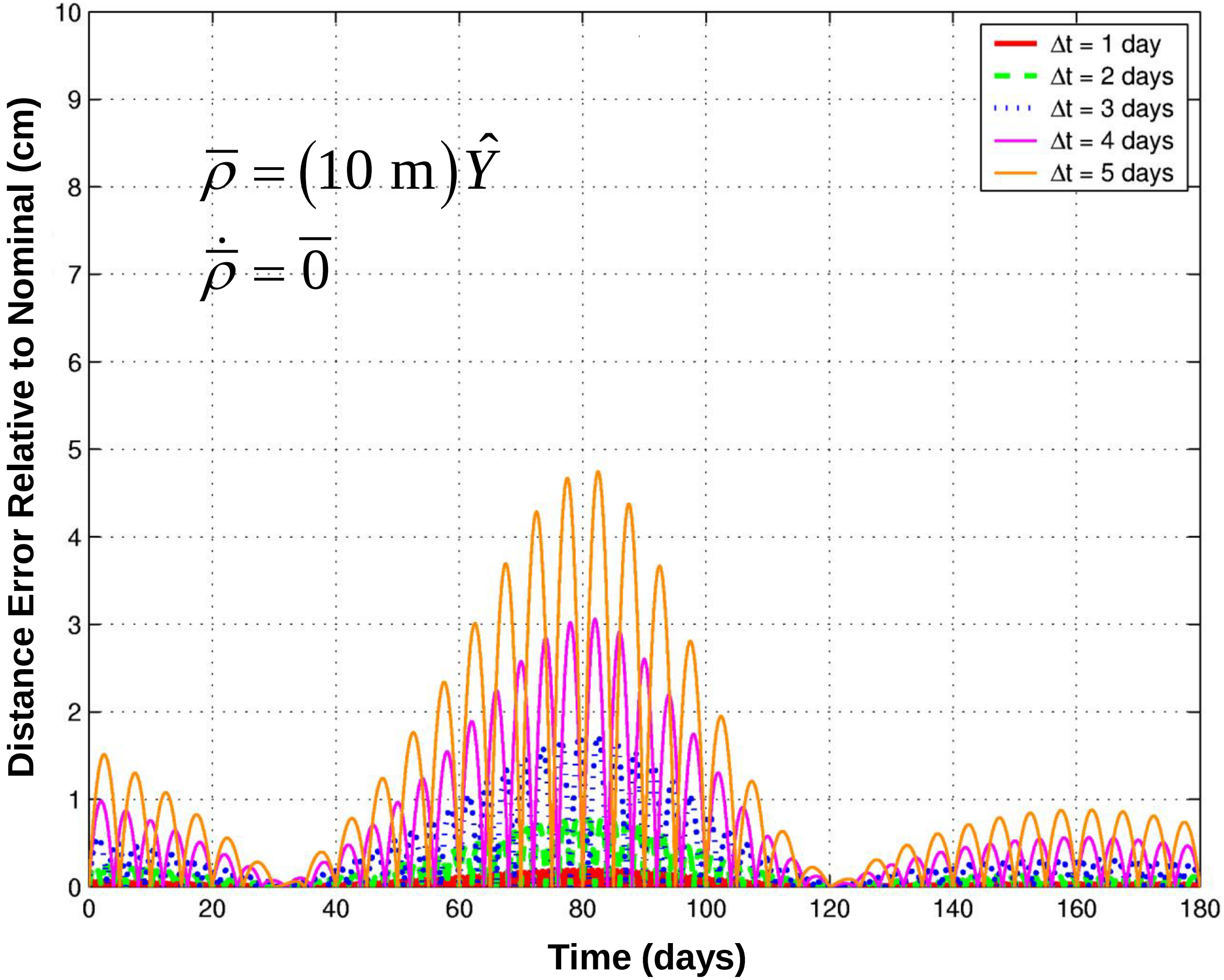

Nominal Relative Dynamics in the Synodic Rotating Frame

# Formation Keeping Cost Variation Along the SEM $L_1$ and $L_2$ Halo Families
## (Configurations Fixed in the RLP Frame)

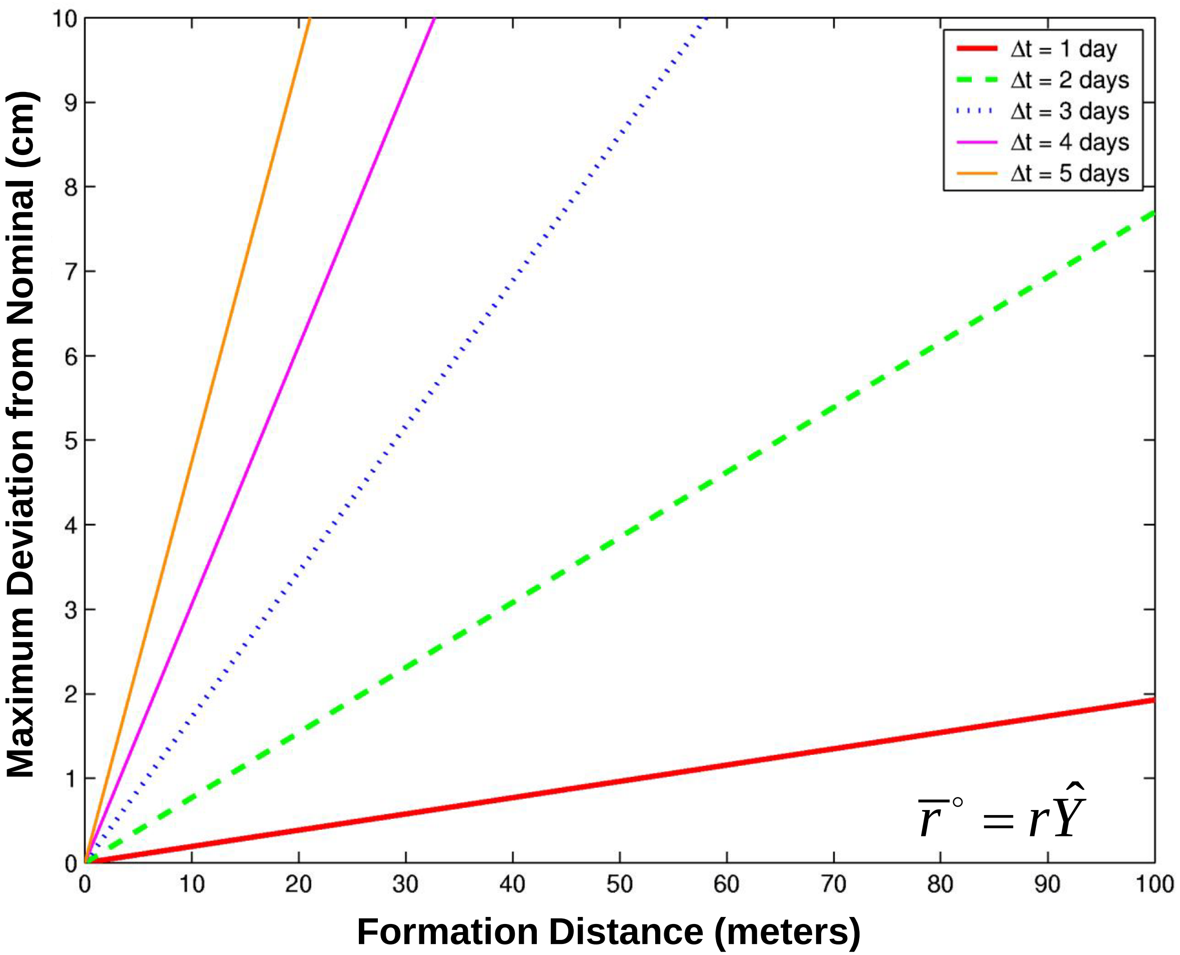


# Discrete vs. Continuous Control

# Discrete Control: Linear Targeter



# Achievable Accuracy via Targeter Scheme



# Continuous Control: LQR vs. Input Feedback Linearization

- LQR for Time-Varying Nominal Motions

$$\dot{\bar{x}}(t)=\begin{bmatrix}\dot{\bar{r}} & \ddot{\bar{r}}\end{bmatrix}^T=\bar{f}\left(t,\bar{x}(t),\bar{u}(t)\right) \quad \rightarrow \bar{x}(0)=\bar{x}_0$$

$$\dot{P}=-A^T(t)P(t)-P(t)A(t)+P(t)B(t)R^{-1}B^T(t)P(t)-Q \rightarrow P\left(t_f\right)=0$$

**Optimal Control Law:**

$$\bar{u}(t)=\overbrace{\bar{u}^{\circ}(t)}^{\text{Nominal Control Input}}+\left\{\underbrace{-R^{-1}B^TP(t)\left(\bar{x}(t)-\bar{x}^{\circ}(t)\right)}_{\text{Optimal Control, Relative to Nominal, from LQR}}\right\}$$

- Input Feedback Linearization (IFL)

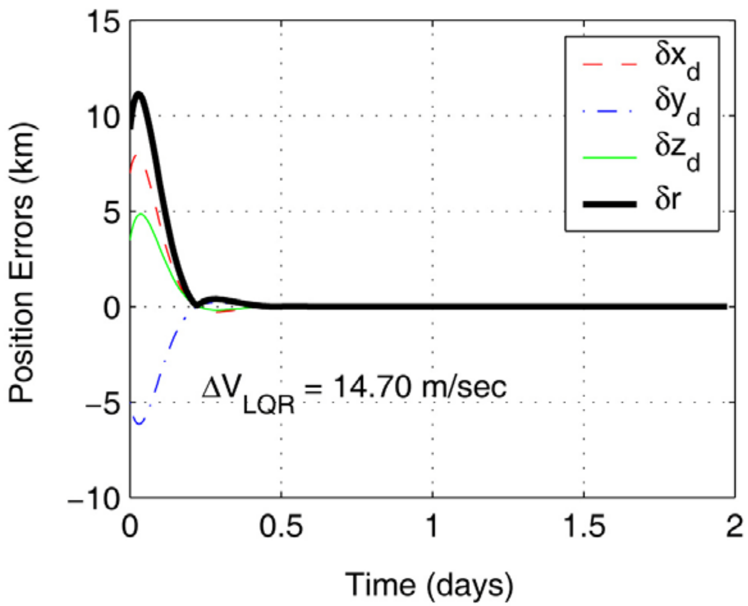
$$\ddot{\bar{r}}(t)=\bar{F}\left(\bar{r}(t)\right)+\bar{u}(t) \quad \Rightarrow \quad \bar{u}(t)=\underbrace{-\bar{F}\left(\bar{r}(t)\right)}_{\text{Anihilate Natural Dynamics}}+\overbrace{\bar{g}\left(\bar{r}(t),\dot{\bar{r}}(t)\right)}^{\text{Desired Dynamic Response}}$$

# Dynamic Response to Injection Error

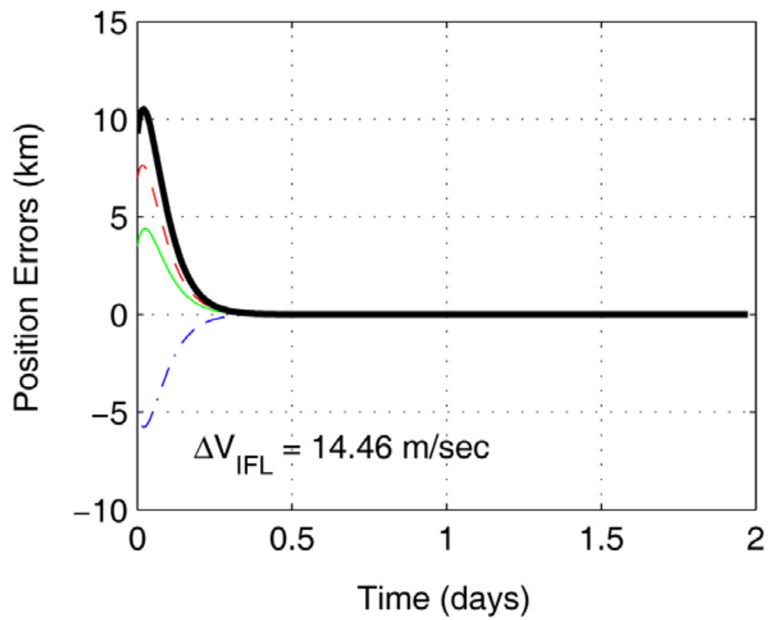
$$\rho = 5000 \text{ km}, \ \xi = 90^\circ, \ \beta = 0^\circ$$

$$\delta\bar{x}(0) = [7 \text{ km} \quad -5 \text{ km} \quad 3.5 \text{ km} \quad 1 \text{ mps} \quad -1 \text{ mps} \quad 1 \text{ mps}]^T$$

**LQR Controller**




**IFL Controller**




Dynamic Response Modeled in the CR3BP
Nominal State Fixed in the Rotating Frame

# Output Feedback Linearization (Radial Distance Control)

## Formation Dynamics

$$\ddot{\bar{r}} = \Delta\bar{f}\left(\bar{r}\right) + \bar{u}\left(t\right) \quad \rightarrow \text{ Generalized Relative EOMs}$$

$$y = l\left(\bar{r}\right) \quad \rightarrow \text{ Measured Output}$$

## Measured Output Response (Radial Distance)

$$\ddot{y} = \frac{d^2 l}{dt^2} = \overbrace{p\left(\bar{r},\dot{\bar{r}}\right) + q\left(\bar{r},\dot{\bar{r}}\right)\bar{u}^T\bar{r}}^{\textbf{Actual Response}} = \overbrace{g\left(\bar{r},\dot{\bar{r}}\right)}^{\textbf{Desired Response}}$$

Scalar Nonlinear Functions of $\bar{r}$ and $\dot{\bar{r}}$

## Scalar Nonlinear Constraint on Control Inputs

$$h\left(\bar{r}\left(t\right),\dot{\bar{r}}\left(t\right)\right) - \bar{u}\left(t\right)^T \bar{r}\left(t\right) = 0$$

# Output Feedback Linearization (OFL)

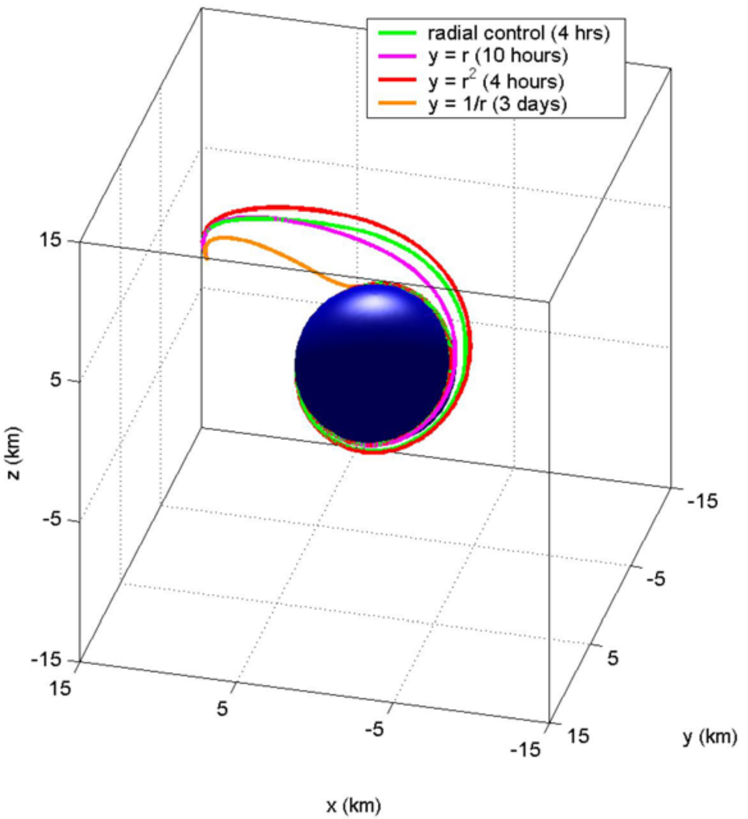
(Radial Distance Control in the Ephemeris Model)

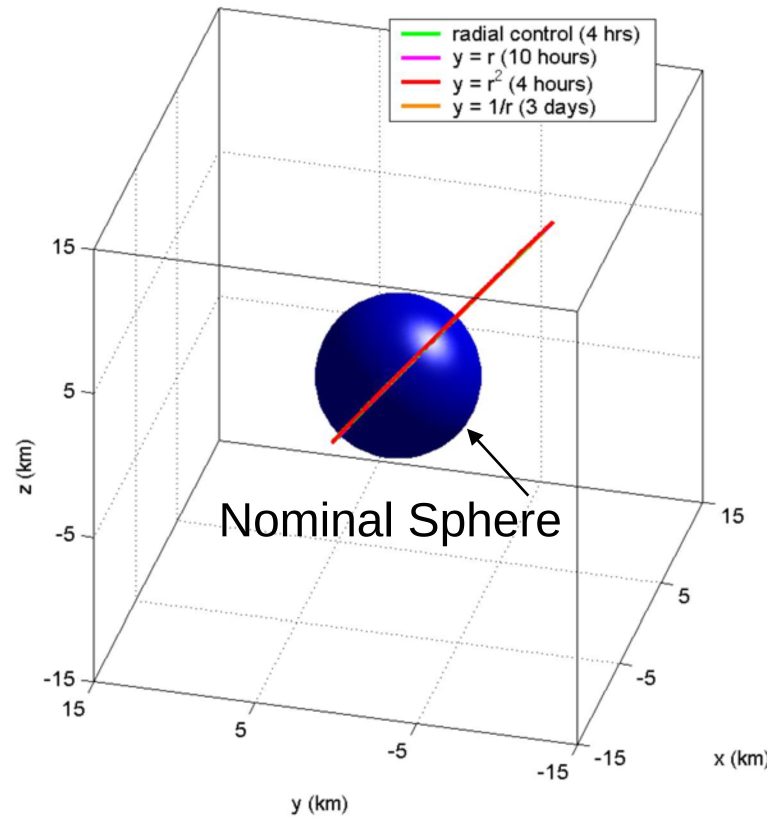
| $y = l(\bar{r}, \dot{\bar{r}})$ | Control Law |
|---|---|
| $r$ | $\bar{u}(t) = \frac{h(\bar{r}, \dot{\bar{r}})}{r}\hat{r}$ Geometric Approach: Radial inputs only |
| $r$ | $\bar{u}(t) = \left\{ \frac{g(\bar{r}, \dot{\bar{r}})}{r} - \frac{\dot{\bar{r}}^T \dot{\bar{r}}}{r^2} \right\} \bar{r} + \left( \frac{\dot{r}}{r} \right) \dot{\bar{r}} - \Delta \bar{f}(\bar{r})$ |
| $r^2$ | $\bar{u}(t) = \left\{ \frac{1}{2} \frac{g(\bar{r}, \dot{\bar{r}})}{r^2} - \frac{\dot{\bar{r}}^T \dot{\bar{r}}}{r^2} \right\} \bar{r} - \Delta \bar{f}(\bar{r})$ |
| $1/r$ | $\bar{u}(t) = \left\{ -rg(\bar{r}, \dot{\bar{r}}) - \frac{\dot{\bar{r}}^T \dot{\bar{r}}}{r^2} \right\} \bar{r} + 3\left( \frac{\dot{r}}{r} \right) \dot{\bar{r}} - \Delta \bar{f}(\bar{r})$ |

- Critically damped output response achieved in all cases
- Total ΔV can vary significantly for these four controllers

# OFL Control of Spherical Formations in the Ephemeris Model

$$\bar{r}(0) = [12 \quad -5 \quad 3] \text{ km} \qquad \dot{\bar{r}}(0) = [1 \quad -1 \quad 1] \text{ m/sec}$$

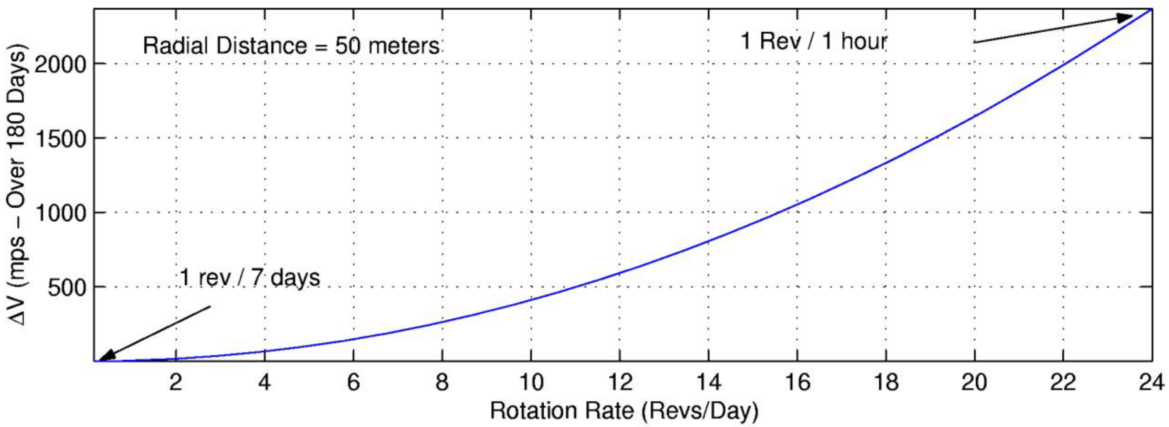


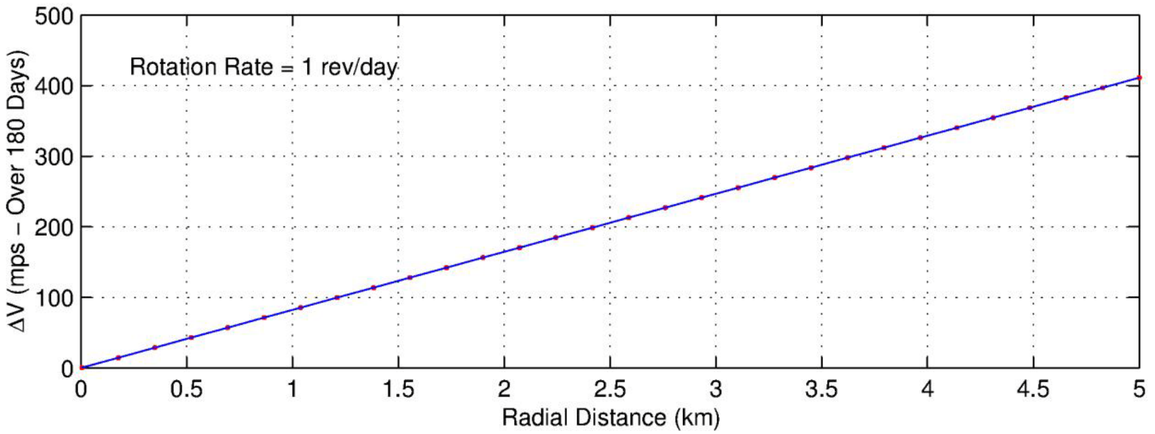

Relative Dynamics as Observed in the Inertial Frame

# OFL Control of Spherical Formations Radial Dist. + Rotation Rate

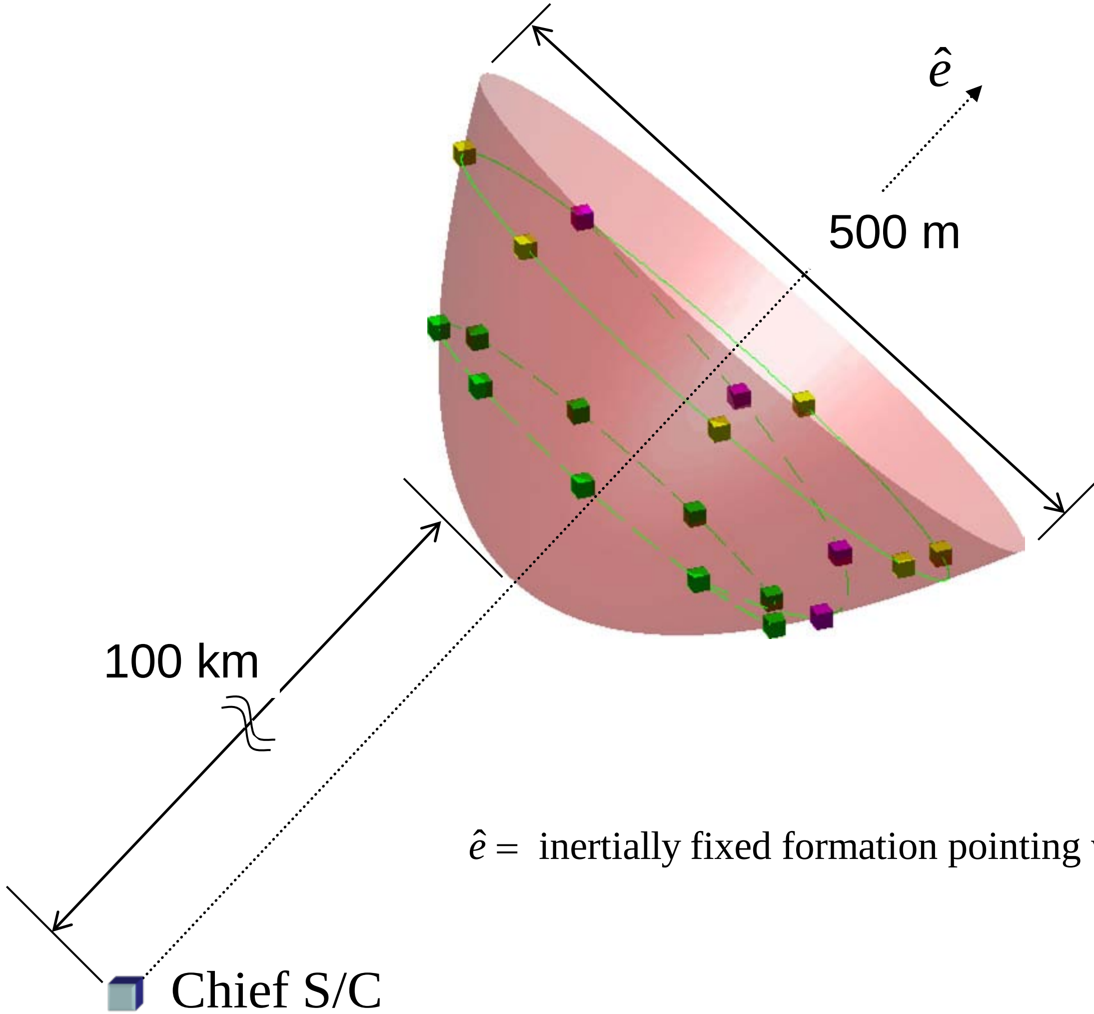
Quadratic Growth in Cost w/ Rotation Rate




Linear Growth in Cost w/ Radial Distance



# Inertially Fixed Formations in the Ephemeris Model



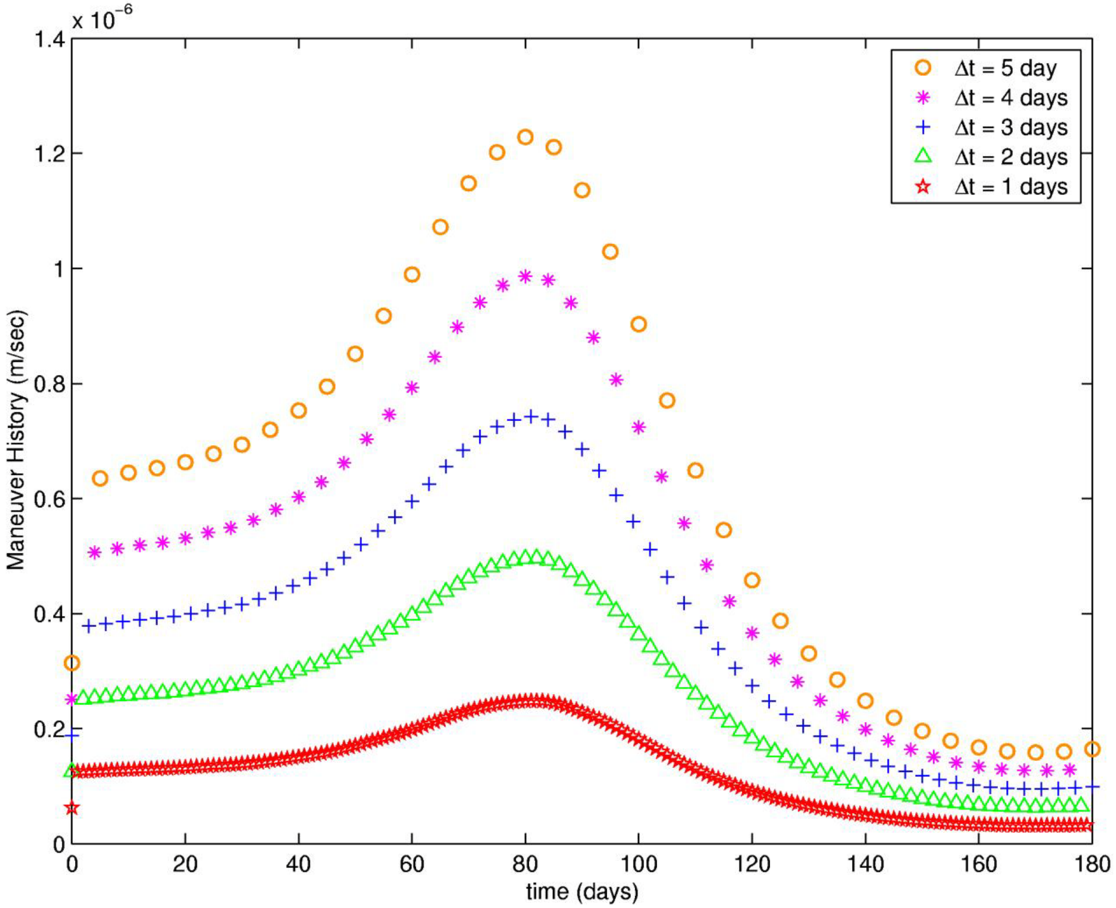

$\hat{e}$ = inertially fixed formation pointing vector (focal line)

# Conclusions

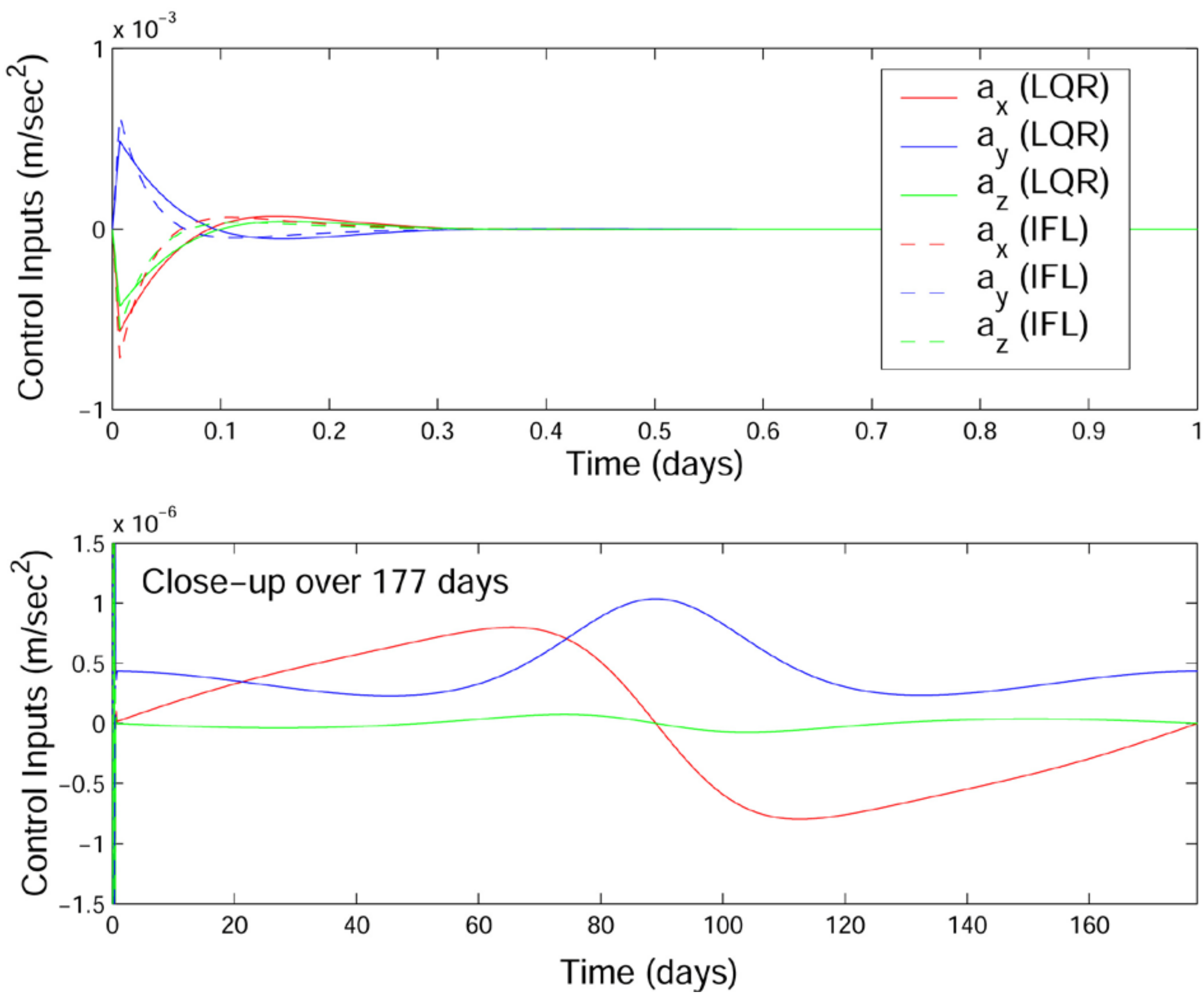
- Continuous Control in the Ephemeris Model:
  - Non-Natural Formations
    - LQR/IFL → essentially identical responses & control inputs
    - IFL appears to have some advantages over LQR in this case
    - OFL → spherical configurations + unnatural rates
    - Low acceleration levels → Implementation Issues
- Discrete Control of Non-Natural Formations
  - Targeter Approach
    - Small relative separations → Good accuracy
    - Large relative separations → Require nearly continuous control
    - Extremely Small $\Delta V$'s ($10^{-5}$ m/sec)
- Natural Formations
  - Nearly periodic & quasi-periodic formations in the RLP frame
  - Floquet controller: numerically ID solutions + stable manifolds

# Backups

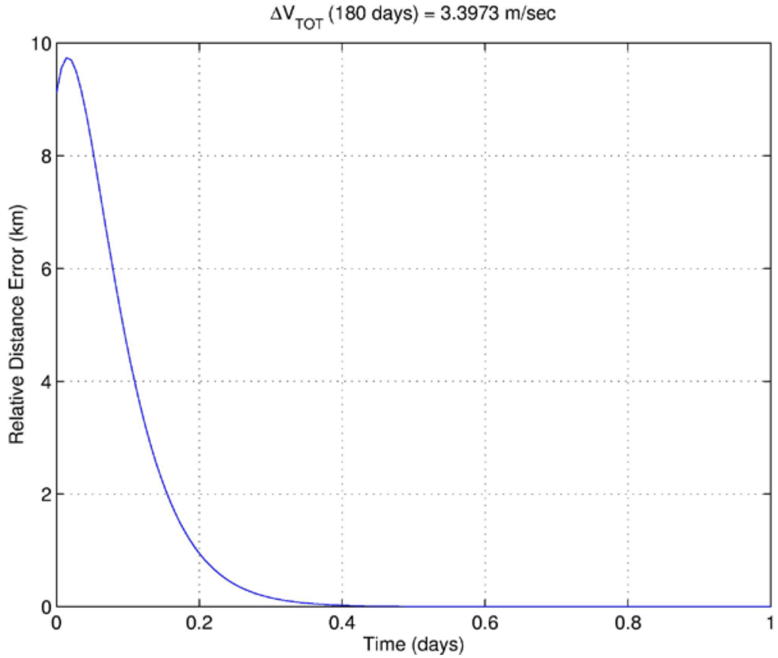
# Targeter Maneuver Schedule

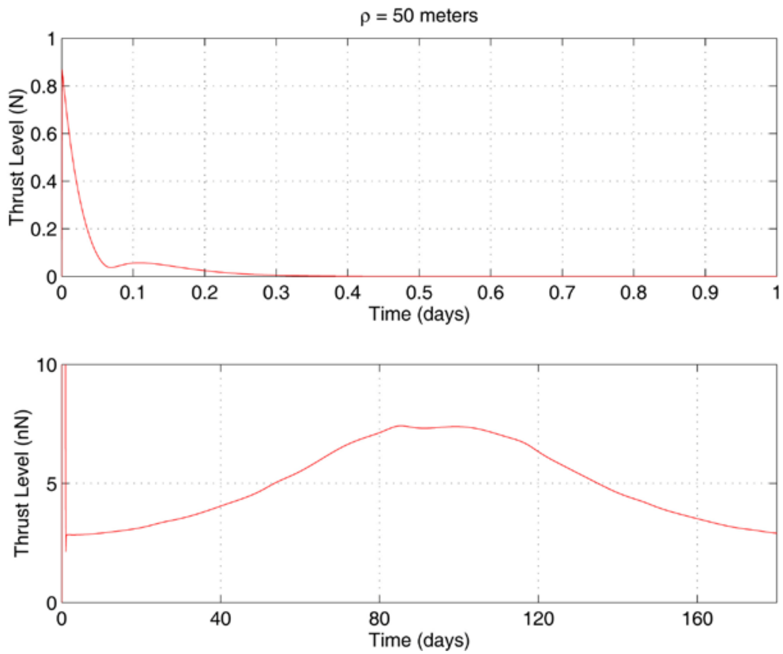


# LQR vs. IFL (CR3BP) Control Accelerations

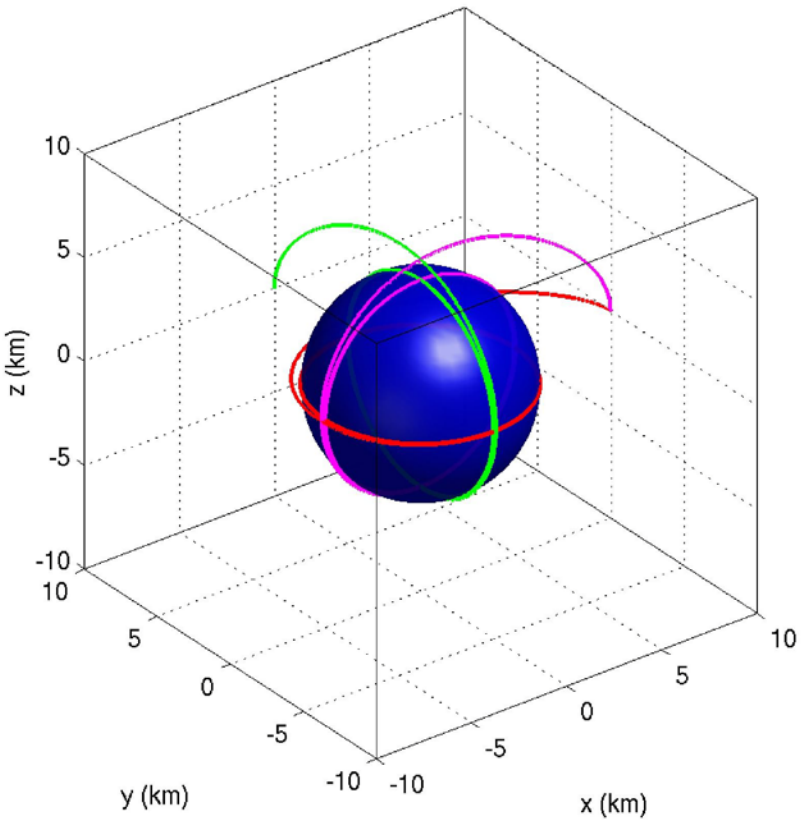


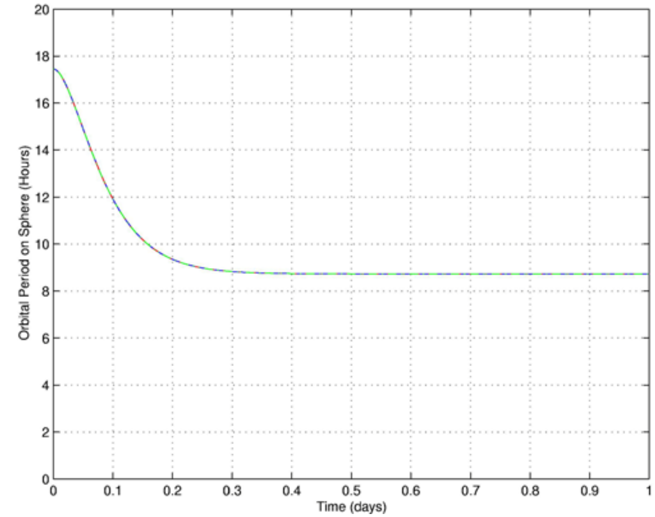
# IFL Response in the Ephemeris Model

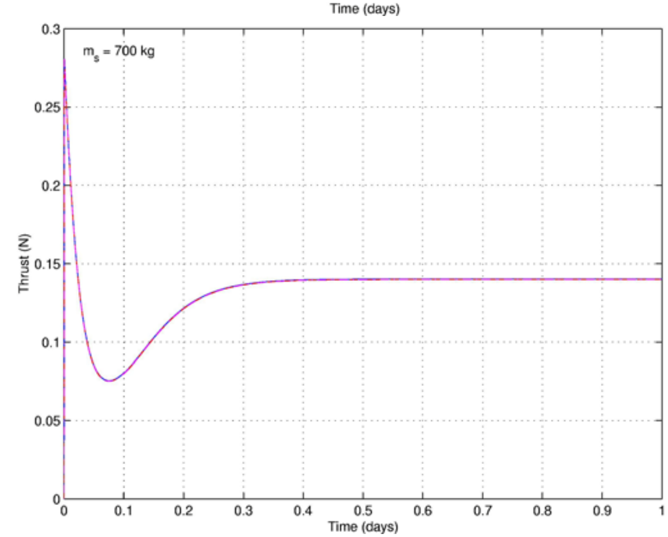


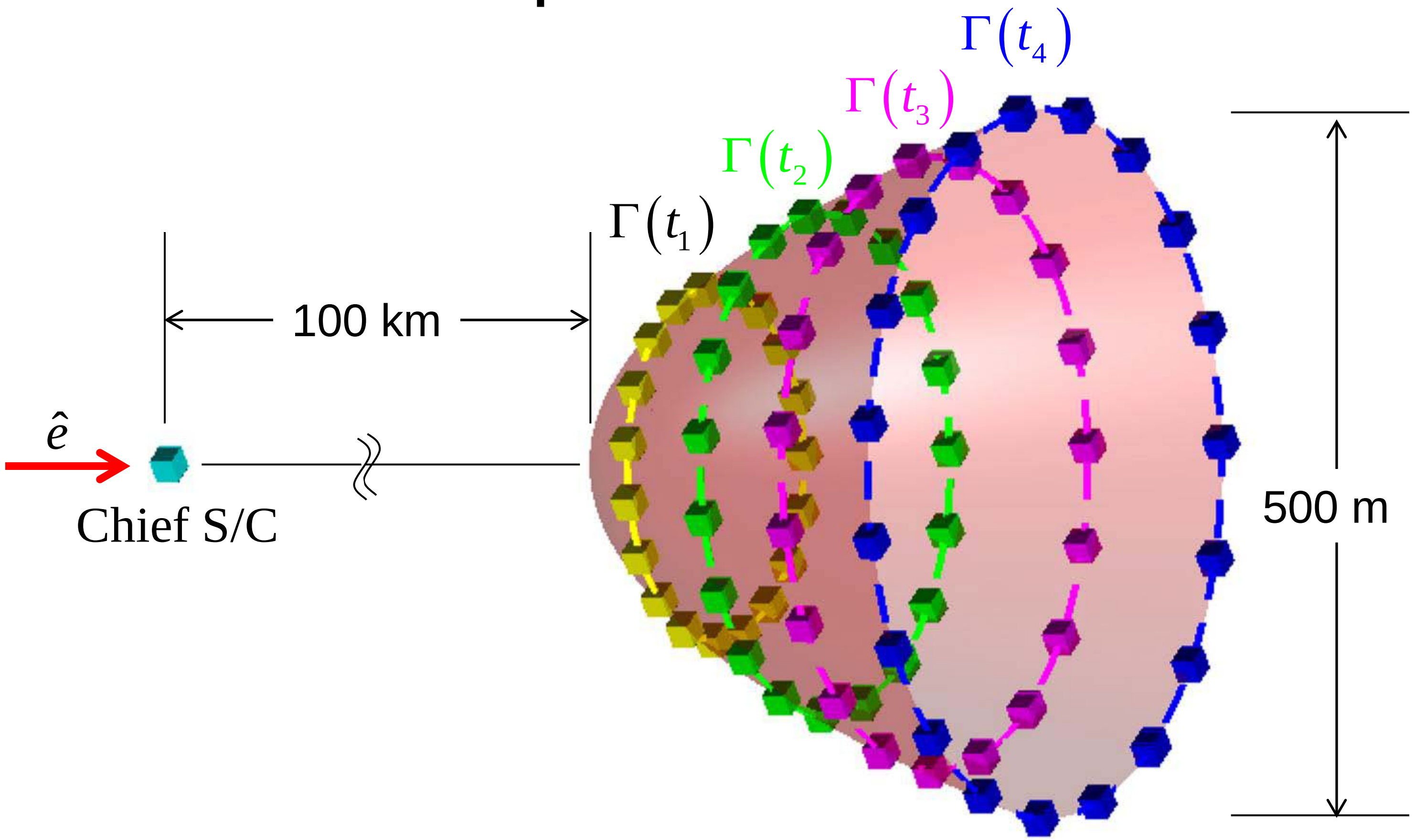
# OFL Control in the Ephemeris Model



# Inertially Fixed Formations in the Ephemeris Model




$\Gamma(t_j)$ = Nominal configuration of deputies (20 s/c) at time $t_j$

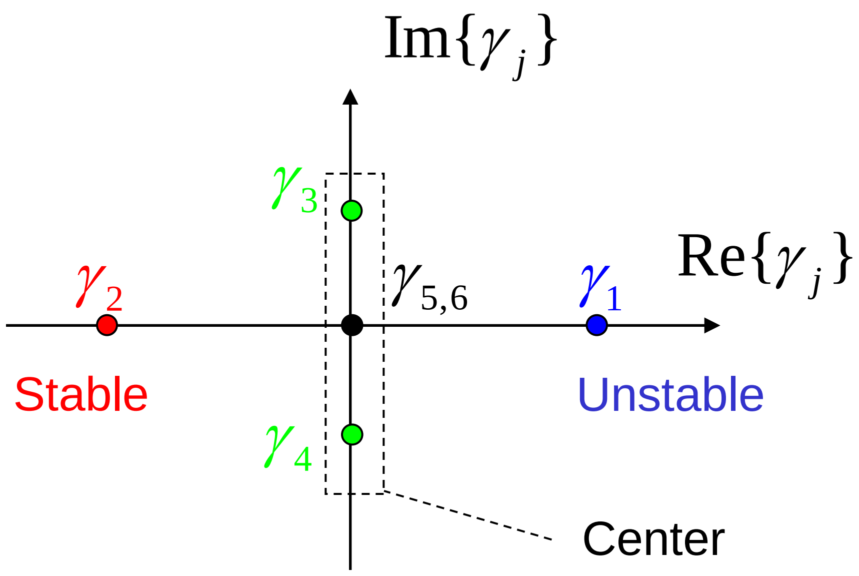
$\hat{e}$ = inertially fixed formation pointing vector (focal line)

# Stability of *T*-Periodic Orbits

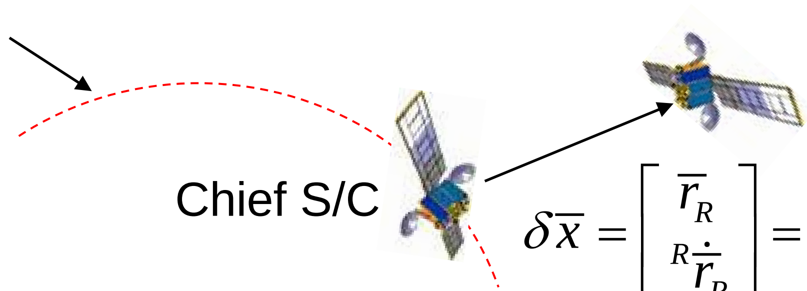
Linear Variational Equation:

$$\delta\overline{x}(t) = \Phi(t,0)\,\delta\overline{x}(0)$$

$\delta\overline{x}(t) \rightarrow$ measured relative to periodic orbit



# Eigenstructure Near Halo Orbit




$$\delta\bar{x} = \begin{bmatrix} \bar{r}_R \\ {}^{R}\dot{\bar{r}}_R \end{bmatrix} = \Phi(t,0)\,\delta\bar{x}(0)$$

Floquet Decomposition of $\Phi(t,0)$:

$$\Phi(t,0) = \{P(t)S\}e^{Jt}\{P(0)S\}^{-1}$$

Floquet Modal Matrix:

$$E(t) = P(t)S = \Phi(t,0)E(0)e^{-Jt}$$

Solution to Variational Eqn. in terms of Floquet Modes:

$$\delta\bar{x}(t) = \sum_{j=1}^{6}\delta\bar{x}_j(t) = \sum_{j=1}^{6}c_j(t)\bar{e}_j(t) = E(t)\bar{c}$$

# Floquet Controller

## (Remove Unstable + 2 Center Modes)

Find $\Delta\bar{v}$ that removes undesired response modes:

$$\sum_{j=1}^{6} \delta\bar{x}_j + \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix} \Delta\bar{v} = \sum_{\substack{j=2,3,4 \\ \text{or} \\ j=2,5,6}} \left(1+\alpha_j\right)\delta\bar{x}_j$$

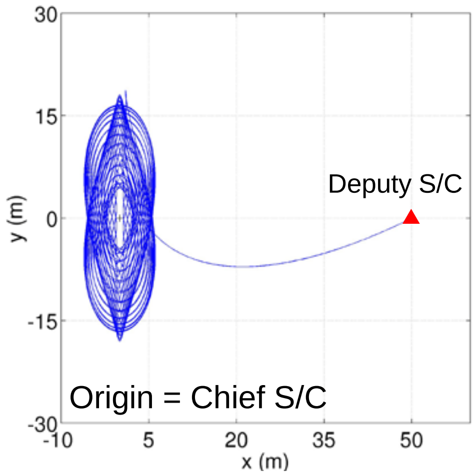
Remove Modes 1, 3, and 4:

$$\begin{bmatrix} \bar{\alpha} \\ \Delta\bar{v} \end{bmatrix} = \begin{bmatrix} \delta\bar{x}_{2\bar{r}} & \delta\bar{x}_{5\bar{r}} & \delta\bar{x}_{6\bar{r}} & 0_3 \\ \delta\bar{x}_{2\bar{v}} & \delta\bar{x}_{5\bar{v}} & \delta\bar{x}_{6\bar{v}} & -I_3 \end{bmatrix}^{-1} \left(\delta\bar{x}_1 + \delta\bar{x}_3 + \delta\bar{x}_4\right)$$

Remove Modes 1, 5, and 6:

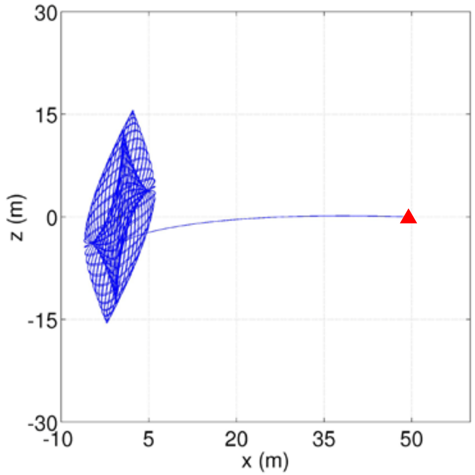
$$\begin{bmatrix} \bar{\alpha} \\ \Delta\bar{v} \end{bmatrix} = \begin{bmatrix} \delta\bar{x}_{2\bar{r}} & \delta\bar{x}_{3\bar{r}} & \delta\bar{x}_{4\bar{r}} & 0_3 \\ \delta\bar{x}_{2\bar{v}} & \delta\bar{x}_{3\bar{v}} & \delta\bar{x}_{4\bar{v}} & -I_3 \end{bmatrix}^{-1} \left(\delta\bar{x}_1 + \delta\bar{x}_5 + \delta\bar{x}_6\right)$$

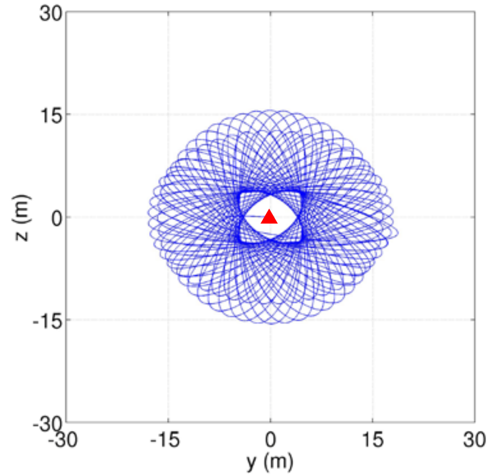
# Deployment into Torus
# (Remove Modes 1, 5, and 6)



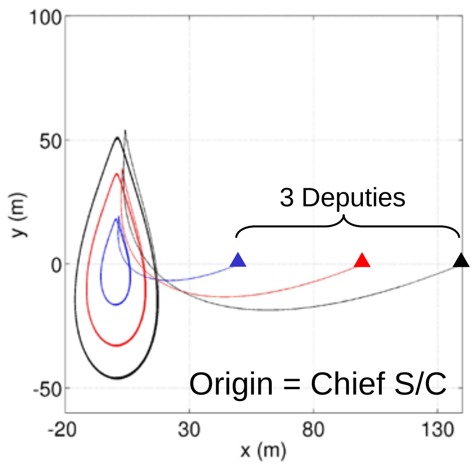

$$\bar{r}(0) = [5 \quad 00 \quad 0] \text{ m}$$

$$\dot{\bar{r}}(0) = [1 \quad -1 \quad 1] \text{ m/sec}$$



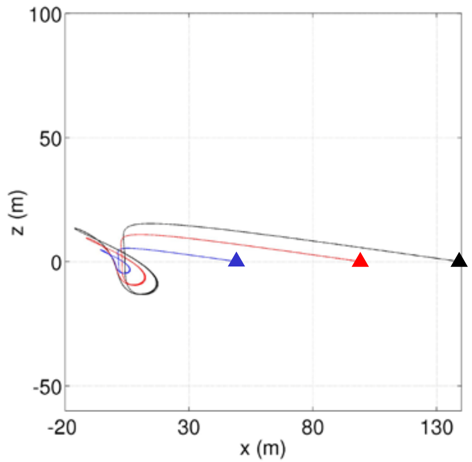



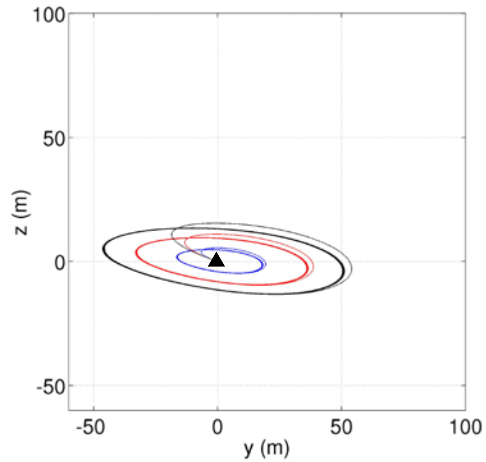
# Deployment into Natural Orbits
## (Remove Modes 1, 3, and 4)



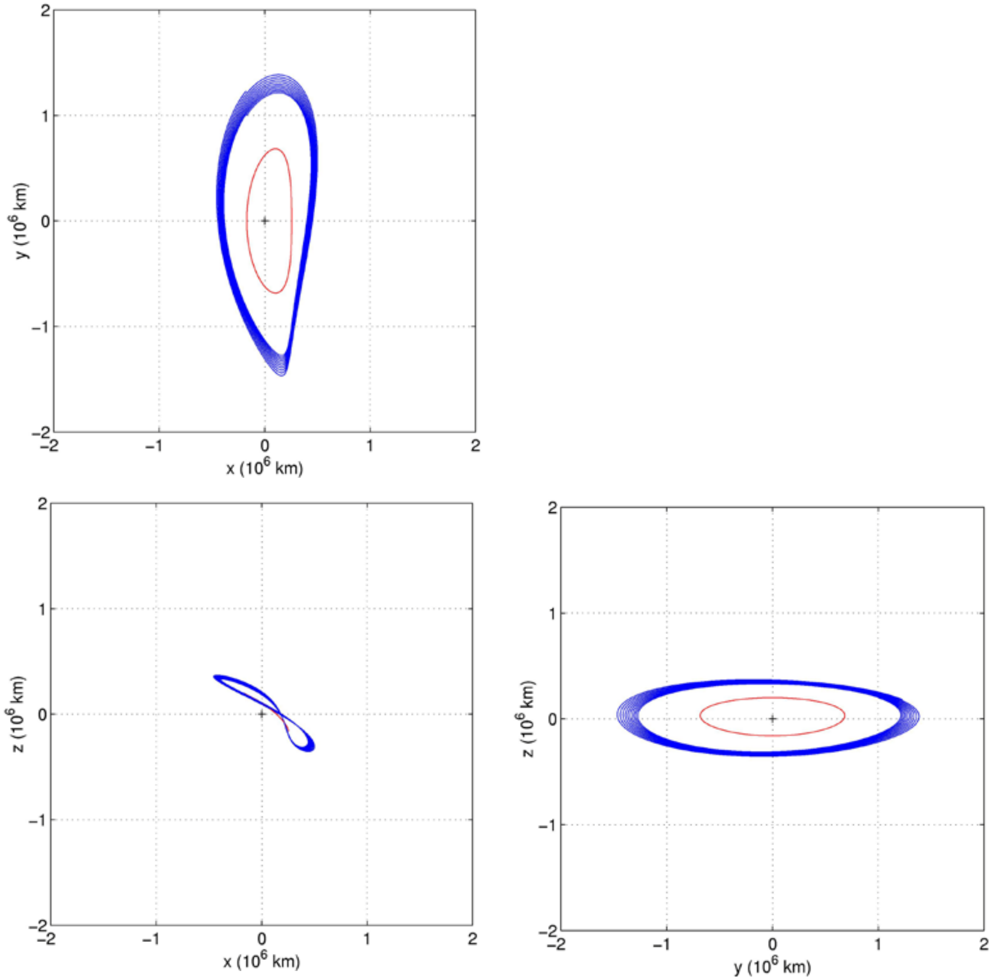

$$\bar{r}(0) = [r_0 \quad 0 \quad 0] \text{ m}$$

$$\dot{\bar{r}}(0) = [1 \quad -1 \quad 1] \text{ m/sec}$$

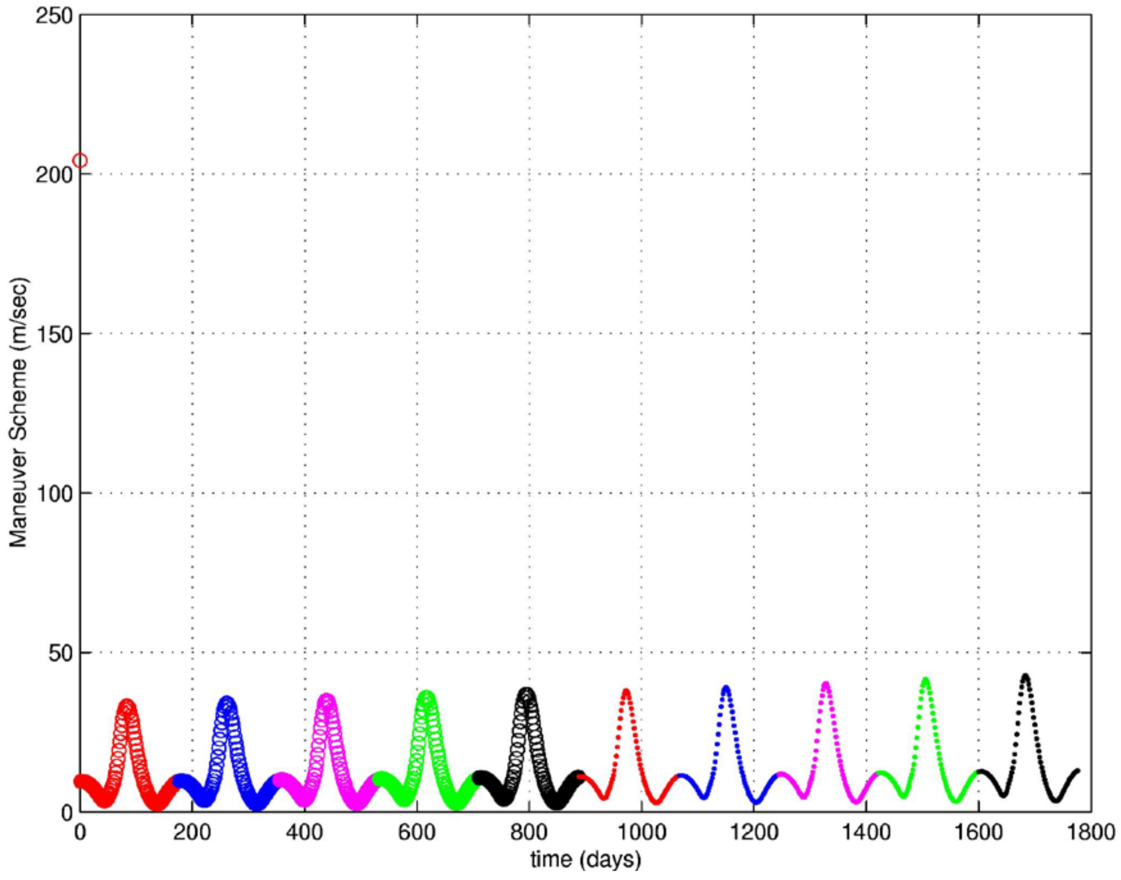


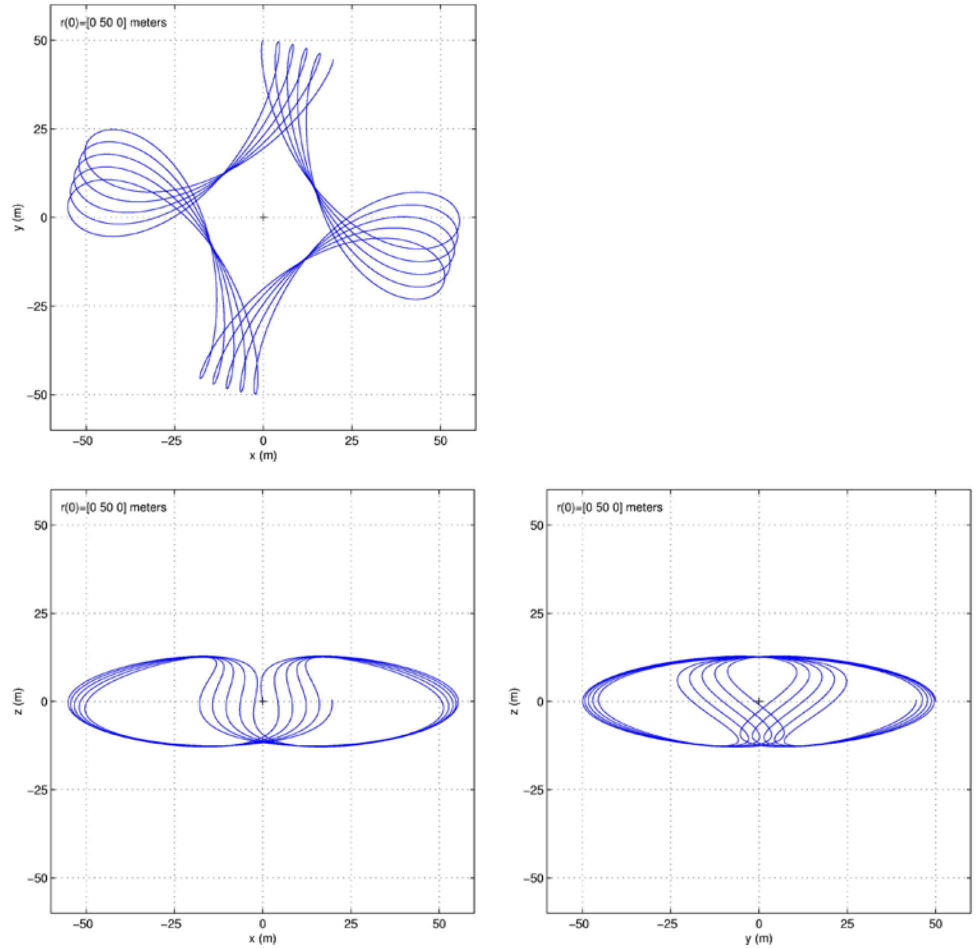
# Floquet Control
# (Large Formations – Example 1)



# Floquet Controller Maneuver Schedule (For Example 1)



# Nearly Periodic Formations (Inertial Perspective)



# Nearly Vertical Formations (Inertial Perspective)