Tracking Control of Nanosatellites with Uncertain Time Varying Parameters

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Motivation

- Spacecraft tracking problem is widely studied.
- Many adaptive control solutions for systems with constant uncertain inertia parameters.
- Limited research in adaptive control of time-varying inertia matrix.
- **Focus of study:** Adaptation mechanism that maintains consistent tracking performance in the face of uncertain time-varying inertia matrix.

Attitude-Tracking Error Dynamics

- Attitude-error dynamics:

$$\dot{q}_{e_0} = -\frac{1}{2}\mathbf{q}_{e_v}^T\omega_e$$
$$\dot{\mathbf{q}}_{e_v} = \frac{1}{2}\left(q_{e_0}\mathbf{I} + [\mathbf{q}_{e_v}\times]\right)\omega_e$$

Angular-velocity tracking error dynamics:

$$\dot{\omega}_e = \mathbf{J}^{-1}\left(-\dot{\mathbf{J}}\omega - [\omega\times]\mathbf{J}\omega + \mathbf{u}\right) + [\omega_e\times]^{\mathcal{B}}\mathbf{C}^{\mathcal{R}}(\mathbf{q}_e)\omega_r - {}^{\mathcal{B}}\mathbf{C}^{\mathcal{R}}(\mathbf{q}_e)\dot{\omega}_r$$

- **Control objective:** Find $\mathbf{u}(t)$ s.t. $\lim_{t\to\infty}\left[\ \mathbf{q}_e, \quad \omega_e\ \right] = 0$ for any $[\mathbf{q}_r(t), \omega_r(t)]$ for all $[\mathbf{q}(0), \omega(0)]$, assuming full feedback $[\mathbf{q}(t), \omega(t)]$ and uncertainty in $\mathbf{J}(t)$.

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Type of Inertia Matrix Considered

- Time-varying inertia matrix of the form

$$\mathbf{J}(t) = \mathbf{J}_o \mathbf{\Psi}(t)$$

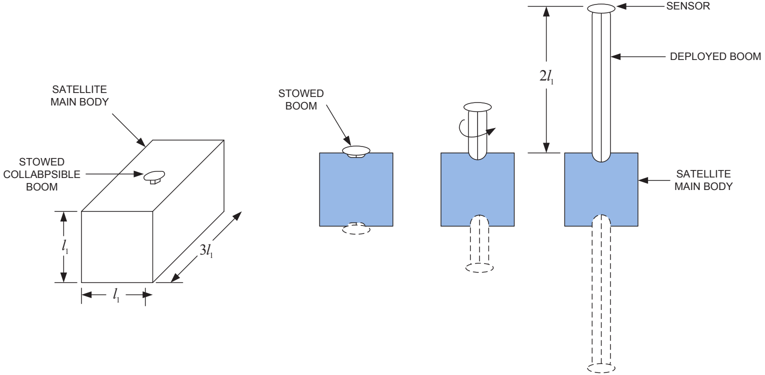
  - $\mathbf{J}_o$: $\left(\mathbf{J}_o > 0,\ \mathbf{J}_o^T = \mathbf{J}_o\right)$ constant, unknown or uncertain
  - $\mathbf{\Psi}(t)$: $\left(\mathbf{\Psi} > 0,\ \mathbf{\Psi}^T = \mathbf{\Psi}\right)$ time-varying, known

- Uncertainty itself is constant, multiplicative

- May be used to model spacecraft undergoing
  1. Thermal Variations
  2. Fuel slosh
  3. Appendage deployment (sensor booms, solar sails, antennas, etc.)

Example of a Time-Varying Inertia Matrix (1/2)

- Consider a spacecraft undergoing boom deployment (e.g., GOES-R spacecraft):
  - Boom extension rate controlled by miniature DC-torque motors
  - Initial mass of prism: m_0
  - Mass of fully extended boom: αm_0, $0 < \alpha < 1$



Example of a Time-Varying Inertia Matrix (2/2)

- Rod length, rod mass, and prism mass are (respectively)

$$r(t) = \frac{2l_1}{\tau}, \quad m_p(t) = \frac{\alpha m_0}{\tau} t, \quad m_c(t) = m_0 - 2m_p(t)$$

- Inertia matrix given by

$$\mathbf{J}_o = \begin{bmatrix} \frac{5}{6} m_0 l_1^2 & 0 & \\ 0 & \frac{5}{6} m_0 l_1^2 & 0 \\ 0 & 0 & \frac{1}{6} m_0 l_1^2 \end{bmatrix}$$

For $0 \leq t \leq \tau$,

$$\boldsymbol{\Psi}(t) = \begin{bmatrix} 1 - 2\frac{\alpha}{\tau} t & 0 & 0 \\ 0 & 1 - \frac{7}{5}\frac{\alpha}{\tau} t + \frac{12}{5}\frac{\alpha}{\tau^2} t^2 + \frac{16}{5}\frac{\alpha}{\tau^3} t^3 & 0 \\ 0 & 0 & 1 + \frac{\alpha}{\tau} t + 12\frac{\alpha}{\tau^2} t^2 + 16\frac{\alpha}{\tau^3} t^3 \end{bmatrix},$$

for $t > \tau$,

$$\boldsymbol{\Psi}(t) = \begin{bmatrix} 1 - 2\alpha & 0 & 0 \\ 0 & 1 - \frac{7}{5}\alpha + \frac{12}{5}\alpha + \frac{16}{5}\alpha & 0 \\ 0 & 0 & 1 + \alpha + 12\alpha + 16\alpha \end{bmatrix}.$$

Control Formulation

- Control method based on the non-certainty equivalence (non-CE) adaptive control results of Seo and Akella (2008)[3].
- Provides superior performance over traditional CE based methods when reference trajectory does not satisfy certain persistence of excitation (PE) conditions.
- Original result treats constant inertia matrix.
- Present investigation modifies original result to handle time-varying inertia matrix of the specific form

$$\mathbf{J}(t) = \mathbf{J}_o \boldsymbol{\Psi}(t).$$

[3] Seo, D. and Akella, M. R., High-Performance Spacecraft Adaptive Attitude-Tracking Control Through Attracting-Manifold Design, *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 4, 2008, pp. 884–891

Non-CE Adaptive Controller

- For problem described by tracking-error equations and inertia matrix $\mathbf{J} = \mathbf{J}_o \boldsymbol{\Psi}(t)$, control input is

$$\begin{aligned}
\mathbf{u} &= \boldsymbol{\Psi}\left(-\mathbf{W}\left(\hat{\boldsymbol{\theta}} + \delta\right) + \mathbf{W}_f \boldsymbol{\Gamma} \mathbf{W}_f^T \left(k_p(\mathbf{q}_{e_v} - \boldsymbol{\omega}_{e_f}) + \boldsymbol{\omega}_e\right)\right) \\
\dot{\hat{\theta}} &= \boldsymbol{\Gamma}\mathbf{W}_f^T \left[(\beta + k_v)\omega_{e_f} + k_p \mathbf{q}_{e_v}\right] - \boldsymbol{\Gamma}\mathbf{W}^T \omega_{e_f} \\
\delta &= \boldsymbol{\Gamma}\mathbf{W}_f^T \omega_{e_f},
\end{aligned}$$

- Regressor matrix

$$\begin{aligned}
\mathbf{W}\boldsymbol{\theta}^* = &-\boldsymbol{\Psi}^{-1}\mathbf{J}_o\dot{\boldsymbol{\Psi}}\omega - \boldsymbol{\Psi}^{-1}[\omega\times]\mathbf{J}_o\boldsymbol{\Psi}\omega + \mathbf{J}_o\left([\omega\times]^{\mathcal{B}}\mathbf{C}^{\mathcal{R}}(\mathbf{q}_e)\omega_r - {}^{\mathcal{B}}\mathbf{C}^{\mathcal{R}}(\mathbf{q}_e)\dot{\omega}_r\right) \\
&+ \mathbf{J}_o\left(k_p\beta\mathbf{q}_{e_v} + k_p\dot{\mathbf{q}}_{e_v} + k_v\boldsymbol{\omega}_e\right),
\end{aligned}$$

- Parameters: $\boldsymbol{\theta}^* = \left[J_{o_{11}}, J_{o_{12}}, J_{o_{13}}, J_{o_{22}}, J_{o_{23}}, J_{o_{33}}\right]^T$.
- Filter variables

$$\begin{aligned}
\dot{\omega}_{e_f} &= -\beta\omega_{e_f} + \omega_e \\
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Numerical Simulations

- Two sets of simulations:
  1. Non-PE reference trajectory
  2. PE reference trajectory

- Simulation features

  - Quantities used to calculate $\mathbf{J}_o$ and $\mathbf{\Psi}$

$$m_0 = 30 \text{ kg}, \quad l = 0.2 \text{ m}, \quad \alpha = 0.1, \quad \tau = 200 \text{ s}$$

  - Uncertain parameter

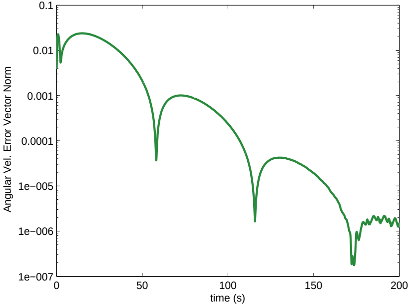
$$\mathbf{J}_o = \begin{bmatrix} 0.2 & 0 & \\ 0 & 0.2 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \longrightarrow \boldsymbol{\theta}^* = [0.2,\ 0,\ 0,\ 0.2,\ 0,\ 1.0]^T$$

  - Initial parameter estimate: $\hat{\boldsymbol{\theta}}(0) + \boldsymbol{\delta}(0) = 1.3\,\boldsymbol{\theta}^*$

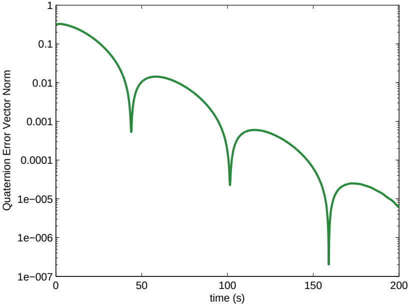
  - Simulation period is $\tau = 200$ seconds.

Non-PE Reference Trajectory (1/3)

- $\boldsymbol{\omega}_r = \left(0.1\cos(t)(1 - e^{0.01t^2}) + (0.08\pi + 0.006\sin(t))te^{-0.01t^2}\right) \cdot [1,\ 1,\ 1]^T$
- Gain values $k_p = 0.08$, $k_v = 0.07$, $\boldsymbol{\Gamma} = \text{diag}\{100, 0.01, 0.01, 200, 0.01, 100\}$.

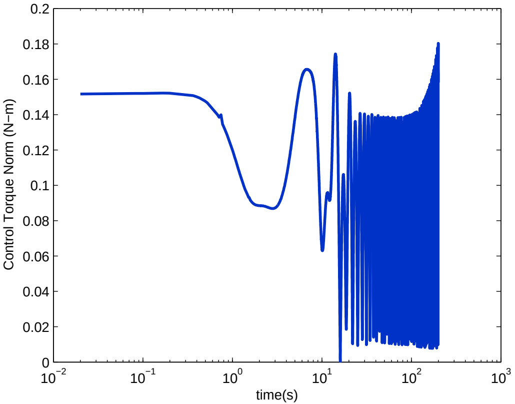


Norm of angular velocity error vector $||\boldsymbol{\omega}_e||$



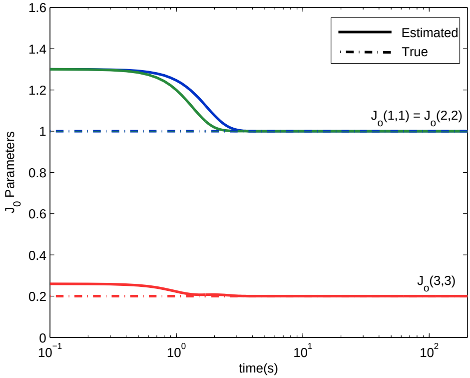
Norm of quaternion error vector $||\mathbf{q}_{e_v}||$

Non-PE Reference Trajectory (2/3)



Norm of control vector $||\mathbf{u}||$

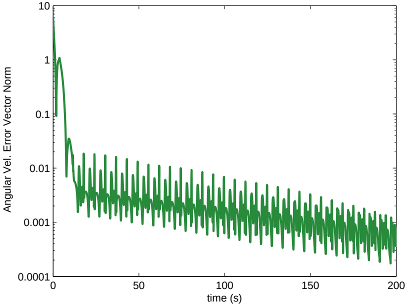
Non-PE Reference Trajectory (3/3)



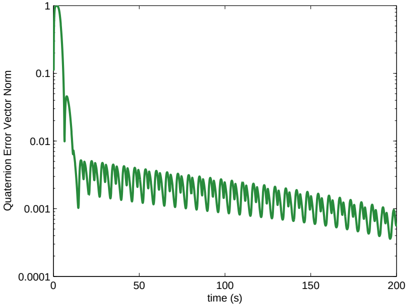
Parameter estimates converge to true values due to additional persistence of excitation introduced by $\mathbf{\Psi}(t)$

PE Reference Trajectory (1/3)

- $\boldsymbol{\omega}_r = \begin{bmatrix} \cos(t)+2 & 5\cos(t) & \sin(t)+2 \end{bmatrix}^T$
- Gain values: $k_p = 0.8$, $k_v = 0.8$, $\boldsymbol{\Gamma} = \text{diag}\{1, 0.001, 0.001, 1, 0.001, 1\}$.

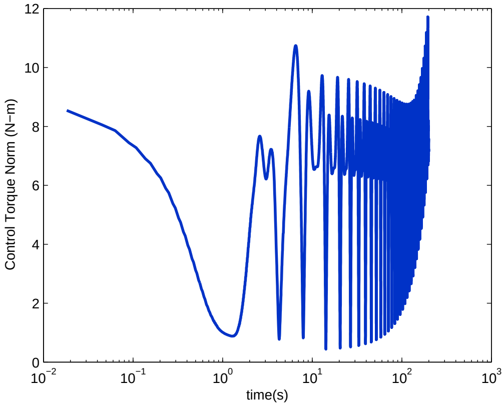


Norm of angular velocity error vector $||\boldsymbol{\omega}_e||$



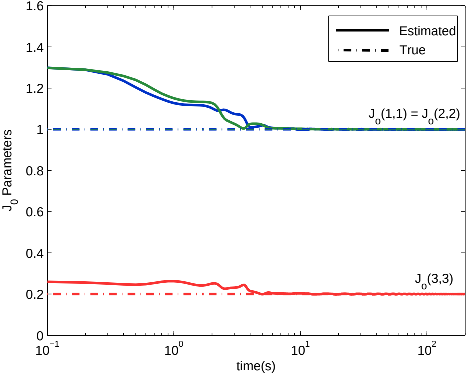
Norm of quaternion error vector $||\mathbf{q}_{e_v}||$

PE Reference Trajectory (2/3)



Norm of control vector $||\mathbf{u}||$

PE Reference Trajectory (3/3)



Parameter estimates converge to true values

Conclusions

- A non-CE adaptive control law employed for spacecraft attitude tracking in the presence of uncertain time-varying inertia matrix.
- Uncertainty has special multiplicative structure.
- Numerical simulations performed for PE and non-PE reference signals.
- Attitude and angular-velocity tracking errors converge to zero.
- Parameter estimates converge to true values even when reference signal is non-PE.

Extra 1: Some Necessary Manipulations

- The following algebraic manipulations are necessary to enable the adaptive control derivation

$$\dot{\boldsymbol{\omega}}_e = \underbrace{-k_p\beta\mathbf{q}_{e_v} - k_p\dot{\mathbf{q}}_{e_v} - k_v\boldsymbol{\omega}_e}_{\text{subtracted term}}$$
$$+ \mathbf{J}_o^{-1}\left(\boldsymbol{\Psi}^{-1}\left(\mathbf{u} - \mathbf{J}_o\dot{\boldsymbol{\Psi}}\boldsymbol{\omega} - [\boldsymbol{\omega}\times]\mathbf{J}_o\boldsymbol{\Psi}\boldsymbol{\omega}\right) - \mathbf{J}_o\boldsymbol{\phi} + \mathbf{J}_o\underbrace{\left(k_p\beta\mathbf{q}_{e_v} + k_p\dot{\mathbf{q}}_{e_v} + k_v\boldsymbol{\omega}_e\right)}_{\text{added term}}\right),$$

  where $\boldsymbol{\phi} = \left([\boldsymbol{\omega}_e\times]^{\mathcal{B}}\mathbf{C}^{\mathcal{R}}(\mathbf{q}_e)\boldsymbol{\omega}_r - {}^{\mathcal{B}}\mathbf{C}^{\mathcal{R}}(\mathbf{q}_e)\dot{\boldsymbol{\omega}}_r\right)$
- $k_p, k_v > 0$ and $\beta = k_p + k_v$
- Note: **Dynamics are unchanged**

Extra 2: Initial Conditions for Simulations

- Initial conditions

$$\mathbf{q}(0) = \begin{bmatrix} 0.9487, & 0.1826, & 0.1826, & 0.18268 \end{bmatrix}^T$$
$$\boldsymbol{\omega}(0) = \begin{bmatrix} 0, & 0, & 0 \end{bmatrix}^T \text{ rad/s}$$
$$\mathbf{q}_r(0) = [1, 0, 0, 0]^T$$
$$\mathbf{W}_f(0) = 0\ ,\ \ \boldsymbol{\omega}_f(0) = \frac{\boldsymbol{\omega}_e(0) + k_p \mathbf{q}_{v_e}(0)}{k_p}$$

- Initial filter-states[1] are $\mathbf{W}_f(0) = 0$ and $\boldsymbol{\omega}_f(0) = \frac{\boldsymbol{\omega}_e(0)+k_p\mathbf{q}_{v_e}(0)}{k_p}$.