Finite Set Control Transcription for Optimal Control Applications

Stuart A. Stanton[1] Belinda G. Marchand[2]

Department of Aerospace Engineering and Engineering Mechanics
The University of Texas at Austin

19th AAS/AIAA Space Flight Mechanics Meeting, February 8-12, 2009
Savannah, Georgia

[1] Capt, USAF; Ph.D. Candidate
[2] Assistant Professor

Outline

System Description

- Hybrid System Dynamics

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{u})$$

- Continuous States

$$\begin{aligned} \boldsymbol{y} &= \begin{bmatrix} y_1 & \cdots & y_{n_y} \end{bmatrix}^T \\ y_i &\in \mathbb{R} \end{aligned}$$

- Discrete Controls

$$\begin{aligned} \boldsymbol{u} &= \begin{bmatrix} u_1 & \cdots & u_{n_u} \end{bmatrix}^T \\ u_i &\in \mathbb{U}_i = \left\{ \tilde{u}_{i,1}, \ldots, \tilde{u}_{i,m_i} \right\} \end{aligned}$$

- Examples
  - Switched Systems
  - Task Scheduling and Resource Allocation Models
  - On-Off Control Systems
  - Control Systems with Saturation Limits

Solving an Optimal Control Problem Numerically

$$
\begin{aligned}
\text{Minimize } \mathcal{J} \quad &= \quad \phi(t_0, \boldsymbol{y}_0, t_f, \boldsymbol{y}_f) + \int_{t_0}^{t_f} L(t, \boldsymbol{y}, \boldsymbol{u})\, dt \\
&\qquad \text{subject to} \\
\dot{\boldsymbol{y}} \quad &= \quad \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{u}), \\
\boldsymbol{0} \quad &= \quad \boldsymbol{\psi}_0(t_0, \boldsymbol{y}_0), \\
\boldsymbol{0} \quad &= \quad \boldsymbol{\psi}_f(t_f, \boldsymbol{y}_f), \\
\boldsymbol{0} \quad &= \quad \boldsymbol{\beta}(t, \boldsymbol{y}, \boldsymbol{u})
\end{aligned}
$$

?

$$
\begin{aligned}
\text{Minimize } \mathcal{J} \quad &= \quad F(\boldsymbol{x}) \\
&\qquad \text{subject to} \\
\boldsymbol{c}(\boldsymbol{x}) \quad &= \quad \left[\boldsymbol{c}_{\dot{y}}^T(\boldsymbol{x}) \ \boldsymbol{c}_{\psi_0}^T(\boldsymbol{x}) \ \boldsymbol{c}_{\psi_f}^T(\boldsymbol{x}) \ \boldsymbol{c}_{\beta}^T(\boldsymbol{x}) \right]^T = \boldsymbol{0}
\end{aligned}
$$

✓

NLP Solver

FSCT Method Overview

- Parameter vector consists only of states and times

$$\boldsymbol{x} = [\cdots \; y_{i,j,k} \; \cdots \; \cdots \; \Delta t_{i,k} \; \cdots \; t_0 \; t_f]^T$$

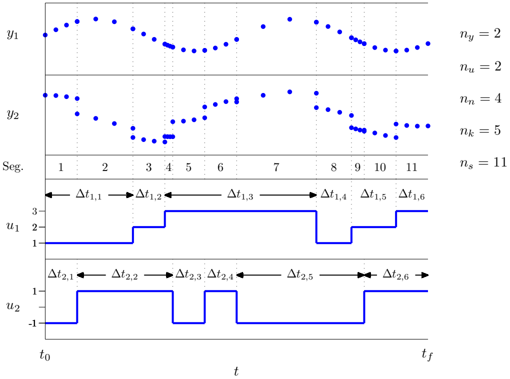
- Control history is completely defined by
  - Pre-specified control sequence
  - Control value time durations, $\Delta t_{i,k}$, between switching points
- Key parameterization factors

n_y Number of States
n_u Number of Controls
n_n Number of Nodes
n_k Number of Knots
n_s Number of Segments ($n_s = n_u n_k + 1$)

FSCT Method Overview

$$\boldsymbol{x} = \left[\cdots\ y_{i,j,k}\ \cdots\ \cdots\ \Delta t_{i,k}\ \cdots\ t_0\ t_f\right]^T$$

$$\begin{aligned} u_1 \in \mathbb{U}_1 &= \{1,2,3\}, \\ u_2 \in \mathbb{U}_2 &= \{-1,1\}. \end{aligned} \qquad \boldsymbol{u}^* = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

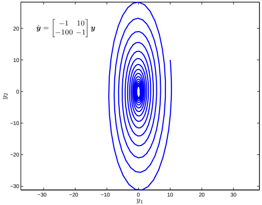


Two Stable Linear Systems

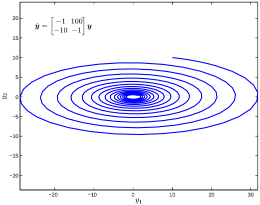
$$\begin{aligned} \dot{\boldsymbol{y}} &= \boldsymbol{f}(\boldsymbol{y}, u) = \boldsymbol{A}_u \boldsymbol{y}, \\ u &\in \{1, 2\}, \end{aligned}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$



(a) $u = 1$



(b) $u = 2$

Figure: Individually Stable Systems

Two Stable Linear Systems

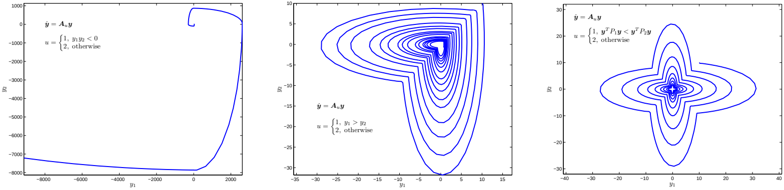
- Several switching laws

(a) Unstable
$$u = \begin{cases} 1, & y_1 y_2 < 0 \\ 2, & \text{otherwise} \end{cases}$$

(b) Stable
$$u = \begin{cases} 1, & y_1 > y_2 \\ 2, & \text{otherwise} \end{cases}$$

(c) Stable
$$u = \begin{cases} 1, & \boldsymbol{y}^T \boldsymbol{P}_1 \boldsymbol{y} < \boldsymbol{y} \boldsymbol{P}_2 \boldsymbol{y} \\ 2, & \text{otherwise} \end{cases}$$

where $\boldsymbol{P}_u \boldsymbol{A}_u + \boldsymbol{A}_u^T \boldsymbol{P}_u = -\boldsymbol{I}$



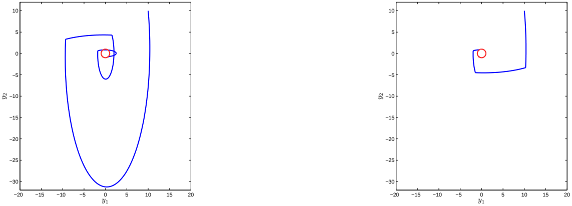
(a) (b) (c)

Figure: Three Switching Laws

Two Stable Linear Systems

- FSCT Optimization

$$\mathcal{J} = F(\boldsymbol{x}) = t_f - t_0$$
$$\boldsymbol{y}_f^T \boldsymbol{y}_f = 1$$
$$u_k^* = \frac{3}{2} + \frac{1}{2}(-1)^k$$



(a) (b)

Figure: FSCT Locally Optimal Switching Trajectories

- Optimization implies the switching law

$$u = \begin{cases} 1, & -\frac{1}{m} \leq \frac{y_2}{y_1} \leq m \\ 2, & \text{otherwise} \end{cases}$$

2-Dimensional Lunar Lander

- Dynamics

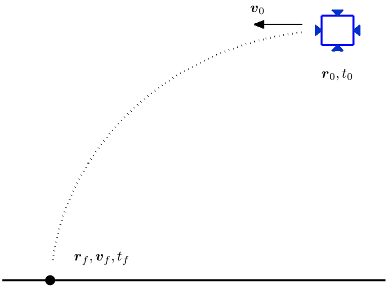
$$\dot{\boldsymbol{y}} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ u_1 \\ -g + u_2 \end{bmatrix},$$

- Controls

$$\begin{aligned} u_1 &\in \{-50,\ 0,\ 50\}\ \text{m/s} \\ u_2 &\in \{-20,\ 0,\ 20\}\ \text{m/s} \end{aligned}$$

- Initial and Final Conditions

$$\begin{aligned} \boldsymbol{r}_0 &= [200\ 15]^T\ \text{km} \\ \boldsymbol{v}_0 &= [-1.7\ 0]^T\ \text{km/s} \\ \boldsymbol{r}_f &= \mathbf{0} \\ \boldsymbol{v}_f &= \mathbf{0} \end{aligned}$$



2-Dimensional Lunar Lander

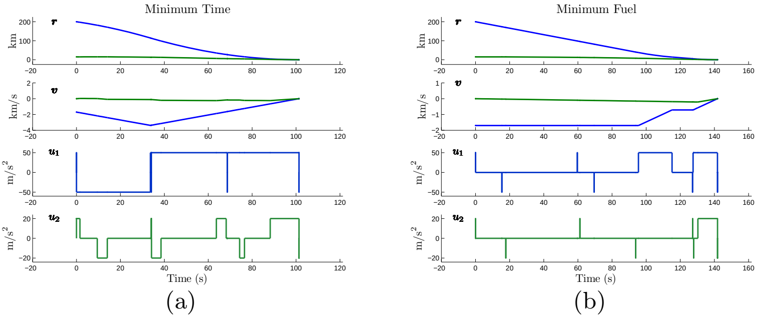
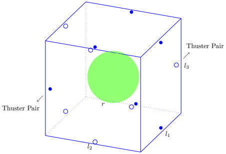


Figure: Optimal Solutions for the Minimum-Time (a) and Minimum-Fuel (b) Lunar Lander Problem

Small Spacecraft Attitude Control: Fixed Thrust

- Fixed thrust cold gas propulsion for arbitrary attitude tracking
  - Reference trajectory defined by ${}^r\boldsymbol{q}^i{}_0$ and ${}^r\boldsymbol{\omega}^i(t)$
- Minimize deviations between body frame and reference frame with minimum propellant mass consumption



$$\mathcal{J} = \beta_1 p_f - \beta_2 m_{p_f}$$

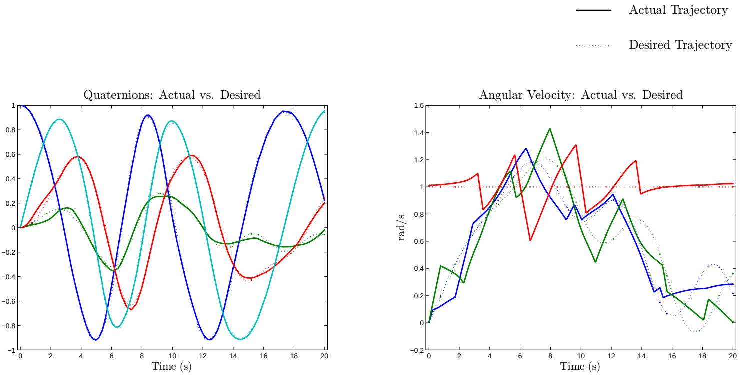
$$p_f - p_0 = \int_{t_0}^{t_f} \dot{p}\, dt = \int_{t_0}^{t_f} \left({}^r\boldsymbol{q}_v{}^b\right)^T \left({}^r\boldsymbol{q}_v{}^b\right)\, dt.$$

$$\dot{\boldsymbol{y}} = \begin{bmatrix} {}^b\dot{\boldsymbol{q}}^i \\ {}^b\dot{\boldsymbol{\omega}}^i \\ \dot{m}_p \\ {}^r\dot{\boldsymbol{q}}^i \\ \dot{p} \end{bmatrix} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{u})$$

$$u_i \in \mathbb{U} = \{-1, 0, 1\}$$

- where u_i indicates for each principal axis whether the positive-thrusting pair, the negative-thrusting pair, or neither is in the on position

Small Spacecraft Attitude Control: Fixed Thrust

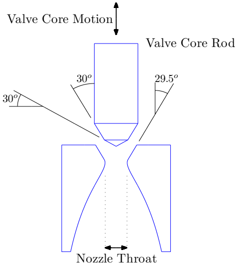


(a) (b)

Figure: Fixed Thrust Attitude Control

Small Spacecraft Attitude Control: Variable Thrust

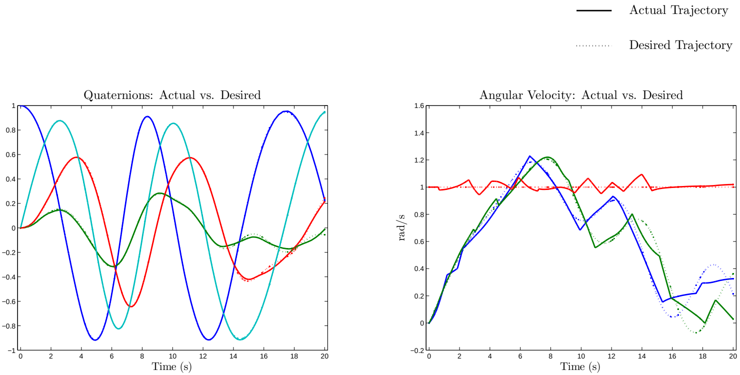
- Variable thrust cold gas propulsion
  - Valve rod modifies nozzle throat area
- Include *additional* states to model variable thrust
  - Resulting dynamics are still hybrid
- States and Controls



$$\boldsymbol{y} = \begin{bmatrix} {}^b\boldsymbol{q}^i \\ {}^b\boldsymbol{\omega}^i \\ m_p \\ \boldsymbol{d} \\ \boldsymbol{v} \\ {}^r\boldsymbol{q}^i \\ p \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{a} \end{bmatrix} \qquad \begin{matrix} w_i \in \{0, 1\} \\ a_i \in \{-1, 0, 1\} \end{matrix}$$

  - w_i indicates whether the i^{th} thruster pair is on or off
  - a_i indicates the acceleration of the valve core rods of the i^{th} thruster pair

Small Spacecraft Attitude Control: Variable Thrust



(a) (b)

Figure: Variable Thrust Attitude Control

Conclusions

- This investigation explores the range of applications of the FSCT method
  - The applicability of the method extends to all engineering disciplines
- FSCT vs. Multiple Lyapunov Functions
  - Optimal control laws may be extracted whose performance exceeds those derived using a Lyapunov argument
- Multiple independent decision inputs managed simultaneously
- Solutions derived via the FSCT method are utilized in conjunction with a hybrid system model predictive control scheme
  - Optimized control schedules can be realized in the context of potential perturbations or other unknowns
- Some continuous control input systems may be more accurately described as systems ultimately relying on discrete decision variables
  - Continuous control variables may often be extended into a set of continuous state variables and discrete inputs