Finite Set Control Transcription for Optimal Control Applications

Stuart A. Stanton
Captain, USAF
Ph.D. Candidate

Department of Aerospace Engineering and Engineering Mechanics
The University of Texas at Austin

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Zermelo Navigation Problem

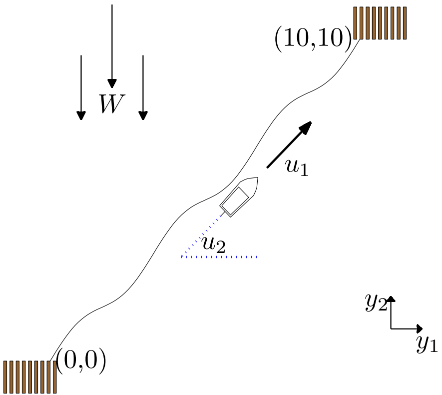


Figure: The Zermelo Navigation Problem

Zermelo Navigation Problem: 1 Segment Solution

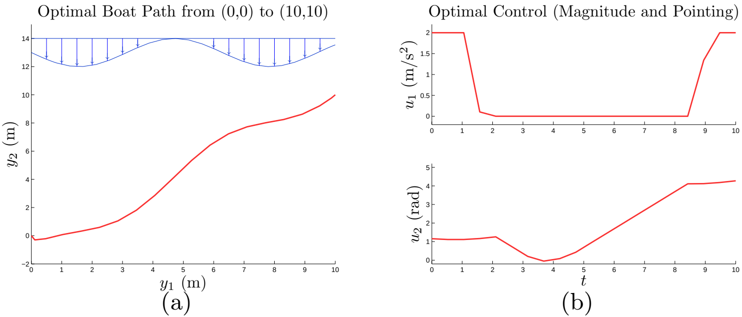


Figure: Optimal Path (a) and Control (b) for the Minimum Acceleration Zermelo Problem

Zermelo Navigation Problem: 3 Segment Solution

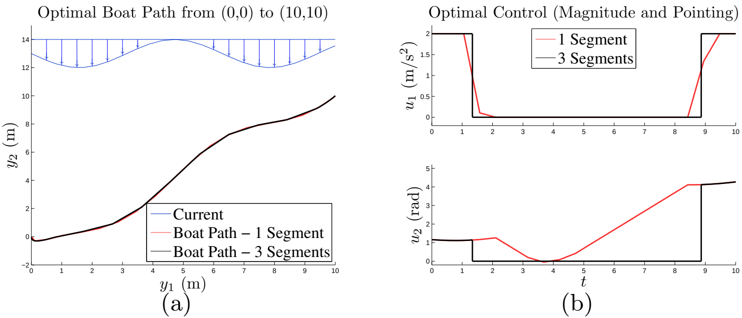


Figure: Optimal Path (a) and Control (b) for the Minimum Acceleration Zermelo Problem

Introduction

- Research Motivation
  - Accomodate realistic actuator constraints into the discovery of optimal solutions
    - Optimality vs. Implementability
  - Realize bang-bang control solutions without ambiguity
- Realm of Hybrid Systems
  - Medical diagnostics
  - Psychology
  - Education
  - Economy
  - Management
  - Sociology
  - Engineering...

System Description

- Hybrid System Dynamics

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{u})$$

- Continuous States

$$\begin{aligned} \boldsymbol{y} &= \begin{bmatrix} y_1 & \cdots & y_{n_y} \end{bmatrix}^T \\ y_i &\in \mathbb{R} \end{aligned}$$

- Discrete Controls

$$\begin{aligned} \boldsymbol{u} &= \begin{bmatrix} u_1 & \cdots & u_{n_u} \end{bmatrix}^T \\ u_i &\in \mathbb{U}_i = \{\tilde{u}_{i,1}, \ldots, \tilde{u}_{i,m_i}\} \end{aligned}$$

- Examples
  - Switched Systems
  - Task Scheduling and Resource Allocation Models
  - On-Off Control Systems
  - Control Systems with Saturation Limits

Solving an Optimal Control Problem Numerically

$$
\begin{aligned}
\text{Minimize } \mathcal{J} \quad &= \quad \phi(t_0, \boldsymbol{y}_0, t_f, \boldsymbol{y}_f) + \int_{t_0}^{t_f} L(t, \boldsymbol{y}, \boldsymbol{u})\, dt \\
&\text{subject to} \\
\dot{\boldsymbol{y}} \quad &= \quad \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{u}), \\
\boldsymbol{0} \quad &= \quad \boldsymbol{\psi}_0(t_0, \boldsymbol{y}_0), \\
\boldsymbol{0} \quad &= \quad \boldsymbol{\psi}_f(t_f, \boldsymbol{y}_f), \\
\boldsymbol{0} \quad &= \quad \boldsymbol{\beta}(t, \boldsymbol{y}, \boldsymbol{u})
\end{aligned}
$$

↘ ?

$$
\begin{aligned}
\text{Minimize } \mathcal{J} \quad &= \quad F(\boldsymbol{x}) \\
&\text{subject to} \\
\boldsymbol{c}(\boldsymbol{x}) \quad &= \quad \left[\boldsymbol{c}_{\dot{y}}^T(\boldsymbol{x})\ \boldsymbol{c}_{\psi_0}^T(\boldsymbol{x})\ \boldsymbol{c}_{\psi_f}^T(\boldsymbol{x})\ \boldsymbol{c}_{\beta}^T(\boldsymbol{x}) \right]^T = \boldsymbol{0}
\end{aligned}
$$

↘ ✓

NLP Solver

FSCT Method Overview

- Parameter vector consists only of states and times

$$\boldsymbol{x} = \left[\cdots \; y_{i,j,k} \; \cdots \; \cdots \; \Delta t_{i,k} \; \cdots \; t_0 \; t_f\right]^T$$

- Control history is completely defined by
  - Pre-specified control sequence
  - Control value time durations, $\Delta t_{i,k}$, between switching points
- Key parameterization factors

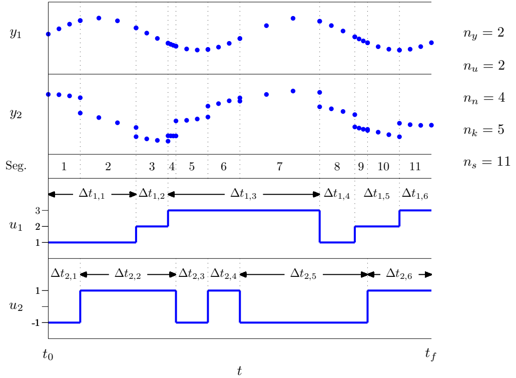
n_y Number of States
n_u Number of Controls
n_n Number of Nodes
n_k Number of Knots
n_s Number of Segments ($n_s = n_u n_k + 1$)

FSCT Method Overview

$$\boldsymbol{x} = [\cdots\ y_{i,j,k}\ \cdots\ \cdots\ \Delta t_{i,k}\ \cdots\ t_0\ t_f]^T$$

$$u_1 \in \mathbb{U}_1 = \{1,2,3\}, \quad u_2 \in \mathbb{U}_2 = \{-1,1\}.$$

$$\boldsymbol{u}^* = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$



FSCT Interface

User Inputs

- System Dynamics

  $$\boldsymbol{f},\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}},\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}},\ \frac{\partial \boldsymbol{f}}{\partial t}$$

- Extra Constraints/ Objectives
- Input File
  - Initial Guess
  - Control Sequence
  - User Options
  - Problem Specifics

↓

FSCT Suite

- Constraint/ Objective Libraries
- Data Manipulation Library
- SNOPT Interface Library

↓

Outputs

- Final Solution

  $$\boldsymbol{x},\ F(\boldsymbol{x}),\ \boldsymbol{c}(\boldsymbol{x}),\ \boldsymbol{\eta}$$

- Convergence Log
- Iteration History

FSCT Process

$$\boldsymbol{x} = [\cdots\ y_{i,j,k}\ \cdots\ \cdots\ \Delta t_{i,k}\ \cdots\ t_0\ t_f]^T$$

↓

Constraint Subroutines

- Initial States
- Initial Time
- Simpson Continuity
- Segment Continuity
- Time
- Final States
- Final Time
- User Defined Constraints

Objective Subroutines

- Control
- Time
- User Defined Objectives

↓

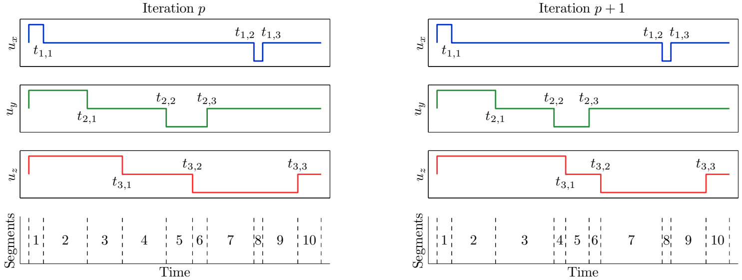
$$F(\boldsymbol{x}) \quad \boldsymbol{c}(\boldsymbol{x})$$

↓

NLP Iteration

Switching Segments and Time for Multiple Independent Controls

- Knots designate switching times in each control axis
- Segments are bounded by switches in any control
- The chronological ordering of knots changes at each iteration of the optimization



Derivative Discontinuities: An Example

- Function

$$f(x_1, x_2) = \min\{x_1, x_2\} = \begin{cases} x_1, & x_1 < x_2, \\ x_1 = x_2, & x_1 = x_2, \\ x_2, & x_1 > x_2. \end{cases}$$

- Derivative

$$\frac{\partial f}{\partial x_1} = \begin{cases} 1, & x_1 < x_2, \\ \text{undefined}, & x_1 = x_2, \\ 0, & x_1 > x_2. \end{cases}$$

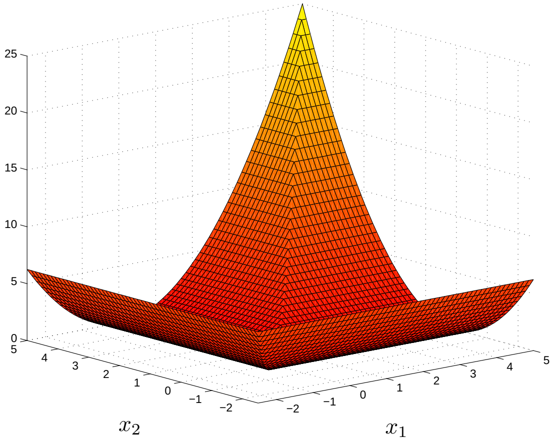
- Candidates for Numerical Implementation

$$\begin{aligned} \text{Forward:} \quad \left.\frac{\partial f}{\partial x_1}\right|_{x_1=x_2} &= \frac{f(x_1+\delta, x_2) - f(x_1, x_2)}{\delta} &= 0, \\ \text{Backward:} \quad \left.\frac{\partial f}{\partial x_1}\right|_{x_1=x_2} &= \frac{f(x_1, x_2) - f(x_1-\delta, x_2)}{\delta} &= 1, \\ \text{Central:} \quad \left.\frac{\partial f}{\partial x_1}\right|_{x_1=x_2} &= \frac{f(x_1+\delta, x_2) - f(x_1-\delta, x_2)}{2\delta} &= \frac{1}{2}. \end{aligned}$$

- Choosing an analytic expression for $\left.\frac{\partial f}{\partial x_1}\right|_{x_1=x_2}$ is equivalent to selecting a finite differencing scheme for numerically evaluated derivatives

Derivative Discontinuities: An Example

$$F(\boldsymbol{x}) = (\min\{x_1, x_2\})^2$$



Derivative Discontinuities: An Example

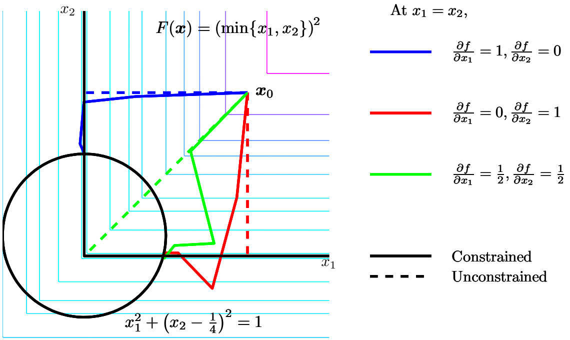


Figure: Effects of Alternative Derivative Definitions on Optimization Path

Implementation of Numerical Derivatives

- An optimizer may calculate finite differences automatically in the absence of analytic derivative expressions
- FSCT method's derivatives may not be effectively evaluated without user intervention
  - Some sophisticated optimizers perform initialization routines to more efficiently calculate numerical derivatives
    - Functions are evaluated from an initial or random point of $\boldsymbol{x}$ to determine the structure of the Jacobian matrix
    - If the initialization routine falsely identifies elements as constant (zero or nonzero), then proper derivatives are not evaluated for future iterations when the knot arrangement is different
- Overcome with user defined procedures to flag *all* potential dependencies (nonzero Jacobian elements) as varying gradients

2-Dimensional Lunar Lander

- Dynamics

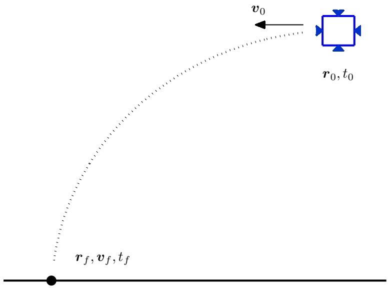
$$\dot{\boldsymbol{y}} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ u_1 \\ -g + u_2 \end{bmatrix}$$

- Controls

$$\begin{aligned} u_1 &\in \{-50,\ 0,\ 50\}\ \mathrm{m/s}^2 \\ u_2 &\in \{-20,\ 0,\ 20\}\ \mathrm{m/s}^2 \end{aligned}$$

- Initial and Final Conditions

$$\begin{aligned} \boldsymbol{r}_0 &= [200\ 15]^T\ \mathrm{km} \\ \boldsymbol{v}_0 &= [-1.7\ 0]^T\ \mathrm{km/s} \\ \boldsymbol{r}_f &= \mathbf{0} \\ \boldsymbol{v}_f &= \mathbf{0} \end{aligned}$$



2-Dimensional Lunar Lander

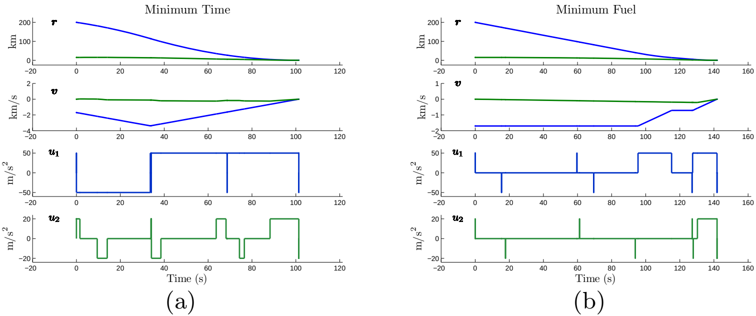
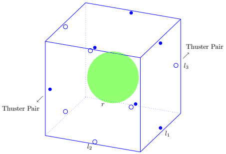


Figure: Optimal Solutions for the Minimum-Time (a) and Minimum-Fuel (b) Lunar Lander Problem

Small Spacecraft Attitude Control: Fixed Thrust

- Fixed thrust cold gas propulsion for arbitrary attitude tracking
  - Reference trajectory defined by ${}^{r}\boldsymbol{q}^{i}{}_{0}$ and ${}^{r}\boldsymbol{\omega}^{i}(t)$
- Minimize deviations between body frame and reference frame with minimum propellant mass consumption



$$\mathcal{J} = \beta_1 p_f - \beta_2 m_{p_f}$$

$$p_f - p_0 = \int_{t_0}^{t_f} \dot{p}\, dt = \int_{t_0}^{t_f} \left({}^{r}\boldsymbol{q}_v{}^{b}\right)^T \left({}^{r}\boldsymbol{q}_v{}^{b}\right) dt.$$

$$\dot{\boldsymbol{y}} = \begin{bmatrix} {}^{b}\dot{\boldsymbol{q}}^{i} \\ {}^{b}\dot{\boldsymbol{\omega}}^{i} \\ \dot{m}_p \\ {}^{r}\dot{\boldsymbol{q}}^{i} \\ \dot{p} \end{bmatrix} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{u})$$

$$u_i \in \mathbb{U} = \{-1, 0, 1\}$$

- where u_i indicates for each principal axis whether the positive-thrusting pair, the negative-thrusting pair, or neither is in the on position

Small Spacecraft Attitude Control: Fixed Thrust

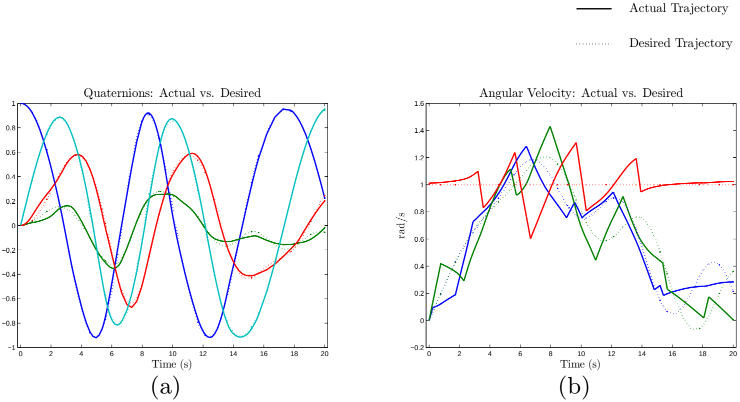
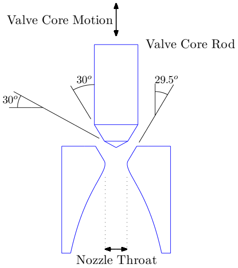


Figure: Fixed Thrust Attitude Control

Small Spacecraft Attitude Control: Variable Thrust

- Variable thrust cold gas propulsion
  - Valve rod modifies nozzle throat area
- Include *additional* states to model variable thrust
  - Resulting dynamics are still hybrid
- States and Controls



$$\boldsymbol{y} = \begin{bmatrix} {}^b\boldsymbol{q}^i \\ {}^b\boldsymbol{\omega}^i \\ m_p \\ \boldsymbol{d} \\ \boldsymbol{v} \\ {}^r\boldsymbol{q}^i \\ p \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{a} \end{bmatrix} \qquad \begin{matrix} w_i \in \{0, 1\} \\ a_i \in \{-1, 0, 1\} \end{matrix}$$

  - w_i indicates whether the i^{th} thruster pair is on or off
  - a_i indicates the acceleration of the valve core rods of the i^{th} thruster pair

Small Spacecraft Attitude Control: Variable Thrust

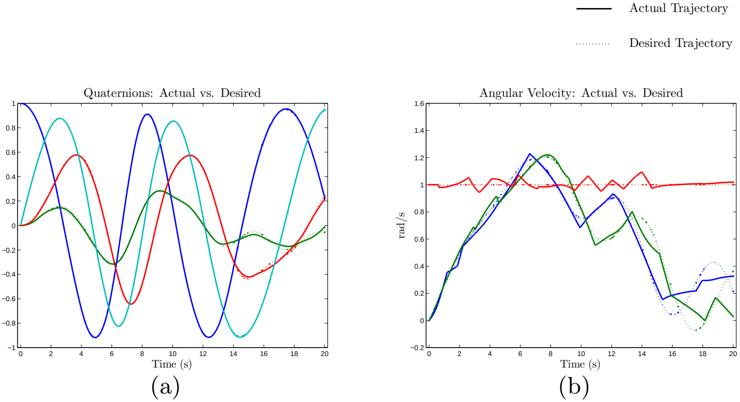


Figure: Variable Thrust Attitude Control

Libration Point Formations: Formation Limitations

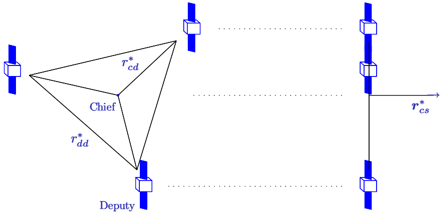


Figure: Formation Pointing

- Fixed size, shape, and orientation of the formation
- Fixed orientation of each member of the formation (deputy spacecraft)

Libration Point Formations: Dynamical Sensitivities

- Previous investigations have focused on unconstrained continuous control solutions
  - Linear and nonlinear; feasible and optimal solutions
  - Non-natural formations require extremely precise control ($<$ nm/s^2 accelerations)
- These controls are impossible to implement with existing actuator technology

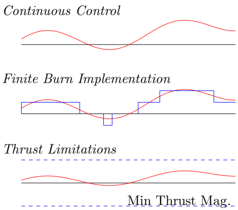


Figure: Implementing a Continuous Control Solution

- Cannot reproduce the fidelity of continuous control
- Continuous control may even be *smaller* than minimum thrust bound

Libration Point Formations: Control Limitations

- Fixed thruster location on each spacecraft body
- Specified thrust acceleration magnitude
  - Based on actuator performance capability

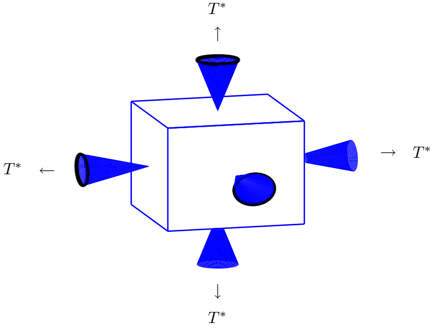


Figure: Spacecraft Body

Costs and Constraints

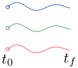
- Constraints
  - Initial time and states specified
  - Final time and formation size and plane specified
    - $r^*_{cd} = 1$ km distance between chief and deputy, $r^*_{dd} = 1.73$ km distance between deputies
    - Specified pointing $\boldsymbol{r}^{*\mathcal{I}}_{cs} = [1\ 0\ 0]$
  - State continuity (differential constraints) by segment
  - State equality across segments (at knots)
- Weighted Costs
  - Minimize thrust
  - Minimize formation size deviations along trajectory
  - Minimize formation plane deviations along trajectory

$$J = w_1 J_1 + w_2 J_2 + w_3 J_3$$
$$F(\boldsymbol{x}) = w_1 F_1(\boldsymbol{x}) + w_2 F_2(\boldsymbol{x}) + w_3 F_3(\boldsymbol{x})$$

Baseline Initial Guess

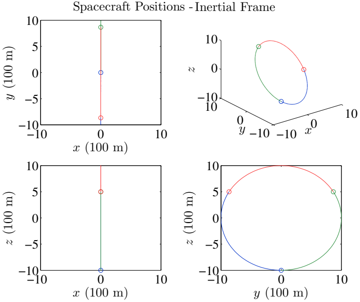
Trajectory Legend

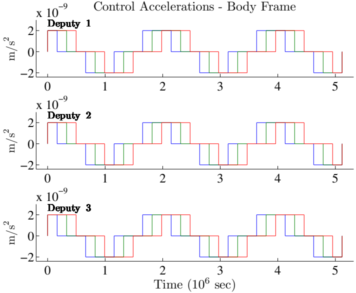
Deputy 1 Trajectory
Deputy 2 Trajectory
Deputy 3 Trajectory



Control Legend

Axis 1 Control (u_x)
Axis 2 Control (u_y)
Axis 3 Control (u_z)





Baseline Solution

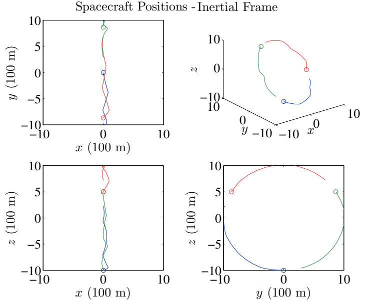
Trajectory Legend

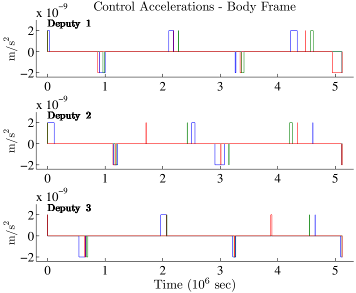
- Deputy 1 Trajectory
- Deputy 2 Trajectory
- Deputy 3 Trajectory

t_0 t_f

Control Legend

- Axis 1 Control (u_x)
- Axis 2 Control (u_y)
- Axis 3 Control (u_z)





Traffic Flow Management

- Density of traffic, ρ, in vehicles/distance unit

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial \tau} = 0,$$

- Velocity of traffic, v, in distance unit/time unit

$$v = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right).$$

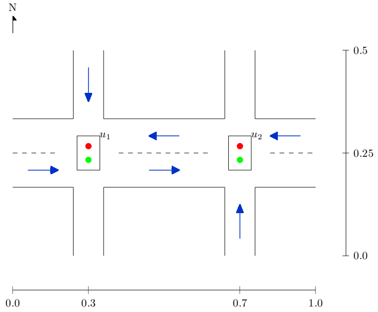
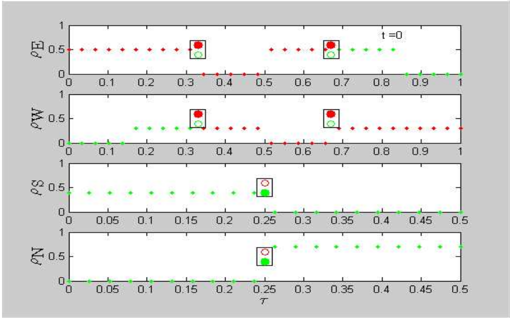


Figure: Traffic Flow Problem: Four Lanes, Two Intersections

Traffic Flow Management



Traffic Flow Management

Conclusions

- Finite Set Control Transcription: This investigation yielded a new metholology for treating *Hybrid Control Systems*
  - Initially designed for *continuous states* and *discrete controls*
  - Extensions are demonstrated for broader classes of hybrid systems
- Future Work
  - Further Development of the Methodology
    - Apply concepts to direct/indirect shooting methods
    - Investigate mesh refinement
    - Explore extended classes of hybrid systems
  - Further Application of the Methodology
    - Further depth in relevant problems, such as *Spacecraft Attitude Control* and *Traffic Management*
    - Petroleum engineering: *Smart Well Technology*
    - Others...

Action Items

- Spelling and Rewording
  - Marchand: Ch 6-7, App A (Ch 5, App B)
  - D'Souza: Ch 1-7
- Transition to Chapter 5 (Libration Point Formations)
- Figure Size and Legibility
- Formatting Considerations

... To be completed before 8 May!

Backup

The Optimization Parameters

$$\boldsymbol{x} = \left[\cdots\ y_{i,j,k}\ \cdots\ \cdots\ \Delta t_{i,k}\ \cdots\ t_0\ t_f\right]^T$$

Array	Description	Dimension	Element
$\boldsymbol{Y}_{n_y,n_n,n_s}$	States by node	$n_y \times n_n \times n_s$	$\boldsymbol{y}_{j,k}$ or $y_{i,j,k}$
$\boldsymbol{U}^*_{n_u,n_k+1}$	Pre-specified controls	$n_u \times (n_k+1)$	$u^*_{i,k}$
$\boldsymbol{U}_{n_u,n_s}$	Controls by segment	$n_u \times n_s$	$u_{i,k}$
$\boldsymbol{U}_{n_u,n_n,n_s}$	Controls by node	$n_u \times n_n \times n_s$	$\boldsymbol{u}_{j,k}$ or $u_{i,j,k}$
$\boldsymbol{T}_{n_n,n_s}$	Node times	$n_n \times n_s$	$t_{j,k}$
$\boldsymbol{T}_{n_u,n_k}$	Knot times	$n_u \times n_k$	$t_{i,k}$
$\Delta\boldsymbol{T}_{n_u,n_k+1}$	Axis durations	$n_u \times (n_k+1)$	$\Delta t_{i,k}$
$\boldsymbol{T}'_{n_s+1}$	Unordered knot times	$0 \ldots n_s$	t'_k
$\boldsymbol{T}_{n_s+1}$	Ordered knot times	$0 \ldots n_s$	t_k
n_s	Number of segments	$n_u n_k + 1$	

$\boldsymbol{x} \to \boldsymbol{Y}_{n_y,n_n,n_s},\ \Delta\boldsymbol{T}_{n_u,n_k+1},\ t_0,\ t_f$	
$t_0,\ \Delta\boldsymbol{T}_{n_u,n_k+1} \to \boldsymbol{T}_{n_u,n_k}$	$t_{i,k} = t_0 + \sum_{\kappa=1}^{k} \boxed{\Delta t_{i,\kappa}}$
$t_0,\ \boldsymbol{T}_{n_u,n_k},\ t_f \to \boldsymbol{T}'_{n_s+1}$ $\boldsymbol{T}'_{n_s+1} \to \boldsymbol{T}_{n_s+1}$	$\left[t'_0\ t'_1\ \cdots\ t'_\kappa\ \cdots\ t'_{n_s-1}\ t'_{n_s}\right] \equiv \left[t_0\ t_{1,1}\ \cdots\ t_{i,k}\ \cdots\ t_{n_u,n_k}\ t_f\right]$
$\boldsymbol{T}_{n_u,n_k},\ \boldsymbol{T}_{n_s+1},\ \boldsymbol{U}^*_{n_u,n_k+1} \to \boldsymbol{U}_{n_u,n_s}$	$u_{i,1} = u^*_{i,1}$ $u_{i,k} = u^*_{i,\kappa_i}$ $u_{i,k+1} = \begin{cases} u^*_{i,\kappa_i+1}, & t_k \equiv t_{i,\kappa_i}, \\ u^*_{i,\kappa_i}, & \text{otherwise} \end{cases}$
$\boldsymbol{U}_{n_u,n_s} \to \boldsymbol{U}_{n_u,n_n,n_s}$	$u_{i,j,k} = u_{i,k}$

Dynamical Constraints Using Simpson Integration Equations

$$\boldsymbol{c}_{\dot{y}}(\boldsymbol{x}) = \left[\boldsymbol{c}_{\dot{y}_{1,1}}^T(\boldsymbol{x}) \ \cdots \ \boldsymbol{c}_{\dot{y}_{j,k}}^T(\boldsymbol{x}) \ \cdots \ \boldsymbol{c}_{\dot{y}_{n_n-1,n_s}}^T(\boldsymbol{x})\right]^T$$

where

$$\begin{aligned}\boldsymbol{c}_{\dot{y}_{j,k}}(\boldsymbol{x}) &= \boldsymbol{y}_{j+1,k} - \boldsymbol{y}_{j,k} \\ &\quad - \frac{h}{6}\left[\boldsymbol{f}\left(t_{j,k}, \boldsymbol{y}_{j,k}, \boldsymbol{u}_{j,k}\right) + 4\boldsymbol{f}\left(t_m, \boldsymbol{y}_m, \boldsymbol{u}_m\right) + \boldsymbol{f}\left(t_{j+1,k}, \boldsymbol{y}_{j+1,k}, \boldsymbol{u}_{j+1,k}\right)\right]\end{aligned}$$

and

$$\begin{aligned} h &= \frac{t_k - t_{k-1}}{n_n - 1}, \\ t_{j,k} &= t_{k-1} + h(j-1), \\ t_m &= \frac{1}{2}\left(t_{j,k} + t_{j+1,k}\right) \end{aligned}$$

with midpoint states and controls

$$\begin{aligned} \boldsymbol{y}_m &= \frac{1}{2}\left(\boldsymbol{y}_{j,k} + \boldsymbol{y}_{j+1,k}\right) + \frac{h}{8}\left(\boldsymbol{f}_{j,k} - \boldsymbol{f}_{j+1,k}\right), \\ \boldsymbol{u}_m &= \frac{1}{2}\left(\boldsymbol{u}_{j,k} + \boldsymbol{u}_{j+1,k}\right) = \boldsymbol{u}_{j,k} = \boldsymbol{u}_{j+1,k}. \end{aligned}$$

Partial Derivatives for the Simpson Integration Equations

$$\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{y}_{j,k}} = -\boldsymbol{I} - \frac{h}{6}\left(\frac{\partial \boldsymbol{f}_{j,k}}{\partial \boldsymbol{y}_{j,k}} + 4\frac{\partial \boldsymbol{f}_m}{\partial \boldsymbol{y}_m}\frac{\partial \boldsymbol{y}_m}{\partial \boldsymbol{y}_{j,k}}\right)$$

$$\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{y}_{j+1,k}} = \boldsymbol{I} - \frac{h}{6}\left(4\frac{\partial \boldsymbol{f}_m}{\partial \boldsymbol{y}_m}\frac{\partial \boldsymbol{y}_m}{\partial \boldsymbol{y}_{j+1,k}} + \frac{\partial \boldsymbol{f}_{j+1,k}}{\partial \boldsymbol{y}_{j+1,k}}\right)$$

$$\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{u}_{j,k}} = -\frac{h}{6}\left(\frac{\partial \boldsymbol{f}_{j,k}}{\partial \boldsymbol{u}_{j,k}} + 4\frac{\partial \boldsymbol{f}_m}{\partial \boldsymbol{u}_m}\frac{\partial \boldsymbol{u}_m}{\partial \boldsymbol{u}_{j,k}}\right)$$

$$\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{u}_{j+1,k}} = -\frac{h}{6}\left(4\frac{\partial \boldsymbol{f}_m}{\partial \boldsymbol{u}_m}\frac{\partial \boldsymbol{u}_m}{\partial \boldsymbol{u}_{j+1,k}} + \frac{\partial \boldsymbol{f}_{j+1,k}}{\partial \boldsymbol{u}_{j+1,k}}\right)$$

$$\begin{aligned}\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial t_{k-1}} = &-\frac{1}{6}\left(\boldsymbol{f}_{j,k} + 4\boldsymbol{f}_m + \boldsymbol{f}_{j+1,k}\right)\frac{\partial h}{\partial t_{k-1}} \\ &- \frac{h}{6}\left[\frac{\partial \boldsymbol{f}_{j,k}}{\partial t_{j,k}}\frac{\partial t_{j,k}}{\partial t_{k-1}} + 4\left(\frac{\partial \boldsymbol{f}_m}{\partial \boldsymbol{y}_m}\frac{\partial \boldsymbol{y}_m}{\partial h}\frac{\partial h}{\partial t_{k-1}} + \frac{\partial \boldsymbol{f}_m}{\partial t_m}\frac{\partial t_m}{\partial t_{k-1}}\right) + \frac{\partial \boldsymbol{f}_{j+1,k}}{\partial t_{j+1,k}}\frac{\partial t_{j+1,k}}{\partial t_{k-1}}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial t_{k}} = &-\frac{1}{6}\left(\boldsymbol{f}_{j,k} + 4\boldsymbol{f}_m + \boldsymbol{f}_{j+1,k}\right)\frac{\partial h}{\partial t_{k}} \\ &- \frac{h}{6}\left[\frac{\partial \boldsymbol{f}_{j,k}}{\partial t_{j,k}}\frac{\partial t_{j,k}}{\partial t_{k}} + 4\left(\frac{\partial \boldsymbol{f}_m}{\partial \boldsymbol{y}_m}\frac{\partial \boldsymbol{y}_m}{\partial h}\frac{\partial h}{\partial t_{k}} + \frac{\partial \boldsymbol{f}_m}{\partial t_m}\frac{\partial t_m}{\partial t_{k}}\right) + \frac{\partial \boldsymbol{f}_{j+1,k}}{\partial t_{j+1,k}}\frac{\partial t_{j+1,k}}{\partial t_{k}}\right]\end{aligned}$$

Partial Derivatives for the Simpson Integration Equations

- Divide parameters into state and time elements: $\boldsymbol{x} = \left[\boldsymbol{x}_y^T \ \boldsymbol{x}_t^T\right]^T$
- Partial Derivatives for State-Parameters

$$\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{x}_{y_\gamma}} = \begin{cases} \dfrac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{y}_{j,k}}, & \boldsymbol{x}_{y_\gamma} \equiv \boldsymbol{y}_{j,k}, \\ \dfrac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{y}_{j+1,k}}, & \boldsymbol{x}_{y_\gamma} \equiv \boldsymbol{y}_{j+1,k}, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

- Partial Derivatives for Time-Parameters

$$\frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial \boldsymbol{x}_t} = \frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial t_{k-1}} \frac{\partial t_{k-1}}{\partial \boldsymbol{x}_t} + \frac{\partial \boldsymbol{c}_{\dot{y}_{j,k}}}{\partial t_k} \frac{\partial t_k}{\partial \boldsymbol{x}_t}.$$

where

$$\frac{\partial t_k}{\partial t_0}, \quad \frac{\partial t_k}{\partial t_f}, \quad \text{and} \quad \frac{\partial t_k}{\partial \Delta t_{i,\kappa}}$$

are determined according to knot ordering

Initial States and Time

- Constraint Function

$$\boldsymbol{c}_{\psi_0}(\boldsymbol{x}) = \begin{bmatrix} y_{1,1,1} - (y_0^*)_1 \\ \vdots \\ y_{i,1,1} - (y_0^*)_i \\ \vdots \\ y_{n_y,1,1} - (y_0^*)_{n_y} \\ t_0 - t_0^* \end{bmatrix}$$

- Jacobian elements

$$\frac{\partial c_{\psi_{0_i}}}{\partial x_\gamma} = \begin{cases} 1, & x_\gamma \equiv y_{i,1,1} \Leftrightarrow \gamma = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\partial c_{\psi_{0_{n_y+1}}}}{\partial x_\gamma} = \begin{cases} 1, & x_\gamma \equiv t_0 \Leftrightarrow \gamma = n_y n_n n_s + n_u(n_k+1)+1, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, n_y$

Segment Continuity Between Knots

- Constraint Function

$$\boldsymbol{c}_s(\boldsymbol{x}) = \begin{bmatrix} y_{1,1,2} & - & y_{1,n_n,1} \\ & \vdots & \\ y_{i,1,k+1} & - & y_{i,n_n,k} \\ & \vdots & \\ y_{n_y,1,n_s} & - & y_{n_y,n_n,n_s-1} \end{bmatrix}$$

- Jacobian elements

$$\frac{\partial c_{s_{n_y(k-1)+i}}}{\partial x_\gamma} = \begin{cases} 1, & x_\gamma \equiv y_{i,1,k+1}, \\ -1, & x_\gamma \equiv y_{i,n_n,k}, \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, n_y$ and $k = 1, \ldots, n_s - 1$

Time

- Constraint Function

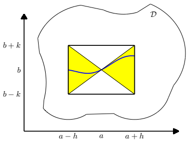
$$\boldsymbol{c}_t(\boldsymbol{x}) = \begin{bmatrix} t_f - t_0 - \sum_{\kappa=1}^{n_k+1} |\Delta t_{1,\kappa}| \\ \vdots \\ t_f - t_0 - \sum_{\kappa=1}^{n_k+1} |\Delta t_{i,\kappa}| \\ \vdots \\ t_f - t_0 - \sum_{\kappa=1}^{n_k+1} |\Delta t_{n_u,\kappa}| \end{bmatrix}$$

- Jacobian elements

$$\frac{\partial c_{t_i}}{\partial x_\gamma} = \begin{cases} 1, & x_\gamma \equiv t_f, \\ -1, & x_\gamma \equiv t_0, \\ & \\ -1, & x_\gamma \equiv \Delta t_{i,k}, \\ 0, & \text{otherwise.} \end{cases}$$

for $i = 1, \ldots, n_u$ and $k = 1, \ldots, n_k + 1$

Existence and Uniqueness



Theorem

Let $\dot{y} = f(t, y, u)$, where u is constant, and $y(a, u) = b$. Suppose that f is continuous in some closed region $\mathcal{D}$ of the t, y plane and hence is bounded. In particular, suppose that

$$|f(t, y, u)| \leq M \text{ over } \mathcal{D}$$

and also that f satisfies a Lipschitz condition in the y argument—that is,

$$|f(t, y_2, u) - f(t, y_1, u)| \leq C|y_2 - y_1|,$$

where the constant C is independent of t or u. Finally, define a rectangle

$$|t - a| \leq h \qquad |y - b| \leq k$$

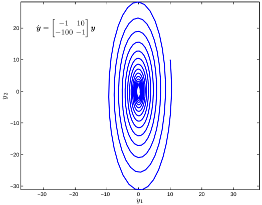
such that $Mh \leq k$. Then $\dot{y} = f(t, y, u)$ has a unique solution $y(t, u)$ in the shaded part of the rectangle.

Two Stable Linear Systems

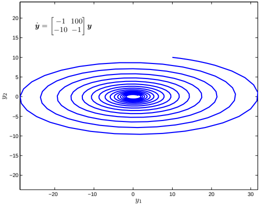
$$\begin{aligned} \dot{\boldsymbol{y}} &= \boldsymbol{f}(\boldsymbol{y}, u) = \boldsymbol{A}_u \boldsymbol{y}, \\ u &\in \{1, 2\}, \end{aligned}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$



(a) $u = 1$



(b) $u = 2$

Figure: Individually Stable Systems

Two Stable Linear Systems

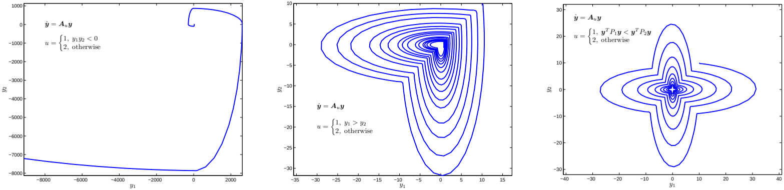
- Several switching laws

(a) Unstable $$u = \begin{cases} 1, & y_1 y_2 < 0 \\ 2, & \text{otherwise} \end{cases}$$

(b) Stable $$u = \begin{cases} 1, & y_1 > y_2 \\ 2, & \text{otherwise} \end{cases}$$

(c) Stable $$u = \begin{cases} 1, & \boldsymbol{y}^T \boldsymbol{P}_1 \boldsymbol{y} < \boldsymbol{y} \boldsymbol{P}_2 \boldsymbol{y} \\ 2, & \text{otherwise} \end{cases}$$

where $\boldsymbol{P}_u \boldsymbol{A}_u + \boldsymbol{A}_u^T \boldsymbol{P}_u = -\boldsymbol{I}$



(a) (b) (c)

Figure: Three Switching Laws

Two Stable Linear Systems

- FSCT Optimization

$$\mathcal{J} = F(\boldsymbol{x}) = t_f - t_0$$
$$\boldsymbol{y}_f^T \boldsymbol{y}_f = 1$$
$$u_k^* = \frac{3}{2} + \frac{1}{2}(-1)^k$$

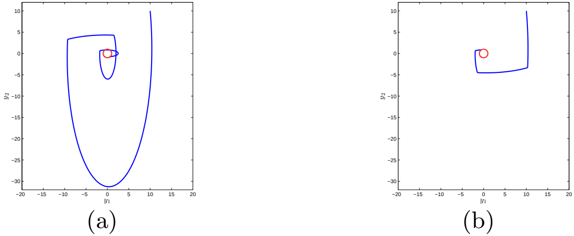


Figure: FSCT Locally Optimal Switching Trajectories

- Optimization implies the switching law

$$u = \begin{cases} 1, & -\frac{1}{m} \le \frac{y_2}{y_1} \le m \\ 2, & \text{otherwise} \end{cases}$$

1-D Lunar Lander

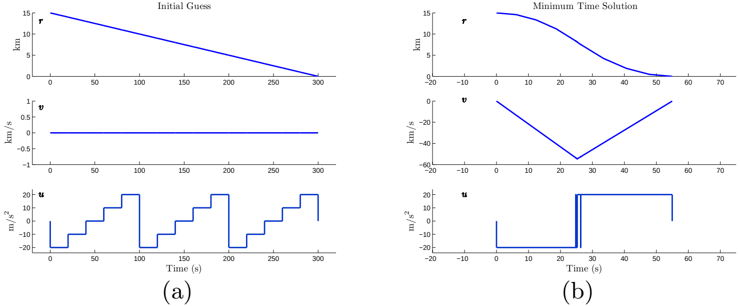


Figure: Initial Guess (a) and Minimum Time Solution (b) for the 1-D Lunar Lander Problem

Equations of Motion

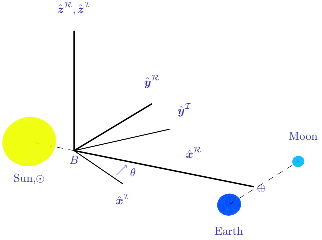


Figure: CR3BP Frame

- Circular Restricted Three-Body Problem (CR3BP) Equations
  - Rotating Frame $\mathcal{R} = \{\hat{\boldsymbol{x}}_{\mathcal{R}}, \hat{\boldsymbol{y}}_{\mathcal{R}}, \hat{\boldsymbol{z}}_{\mathcal{R}}\}$,
- Chief spacecraft lies on a natural trajectory

$$\dot{\boldsymbol{y}}_c = \tilde{\boldsymbol{f}}(\boldsymbol{y}_c, \boldsymbol{u}_c) = \tilde{\boldsymbol{f}}(\boldsymbol{y}_c, \boldsymbol{0})$$

- The lth deputy spacecraft measured relative to the chief

$$\begin{aligned} \boldsymbol{y}_{cd_l} &\equiv \boldsymbol{y}_{d_l} - \boldsymbol{y}_c \\ \dot{\boldsymbol{y}}_{cd_l} &= \boldsymbol{f}(t, \boldsymbol{y}_{cd_l}, \boldsymbol{u}_{d_l}) \end{aligned}$$

Reference Halo Orbit

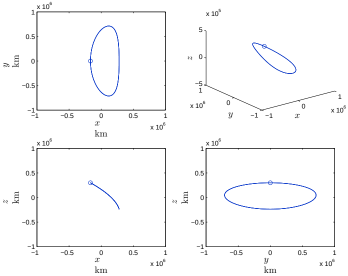


Figure: Reference Halo Orbit for Chief Spacecraft with Origin at L_1

- Chief trajectory is a Halo Orbit about L_1 (approx. 148×10^6 km in the $\hat{\boldsymbol{x}}_{\mathcal{R}}$)
  - At epoch, chief is at northern most point (300,000 km in $\hat{\boldsymbol{z}}_{\mathcal{R}}$)

Impacts of Fixed Spacecraft Orientation

- A traditional finite burn formulation specifies thrust (acceleration) magnitude, but not direction
  - Assumes spacecraft can re-orient to deliver required thrust vector
  - Control space $\mathcal{U}_1$: $\boldsymbol{u}^T\boldsymbol{u} = (T^*)^2$
- If spacecraft orientation is predetermined (according to other mission requirements)
  - Actuator configuration must provide 3-axis maneuverability
    - Assume thrusters are located on principal axes of body frame $\mathcal{B} \equiv \{\hat{\boldsymbol{x}}_\mathcal{B}, \hat{\boldsymbol{y}}_\mathcal{B}, \hat{\boldsymbol{z}}_\mathcal{B}\}$
  - Control space $\mathcal{U}_2$: $u_i(u_i - T^*)(u_i + T^*) = 0, i = \hat{\boldsymbol{x}}_\mathcal{B}, \ldots, \hat{\boldsymbol{z}}_\mathcal{B}$

Fixed spacecraft orientation leads to discrete optimization, which gradient-type NLP algorithms cannot support.

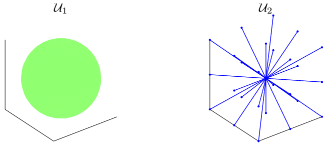
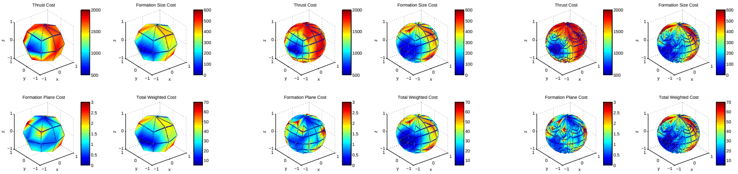


Figure: Control Spaces (a) $\mathcal{U}_1$ (Orientation Free), and (b) $\mathcal{U}_2$ (Orientation Fixed)

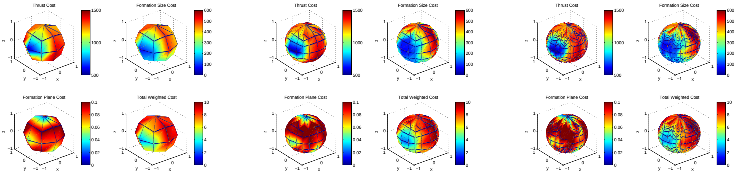
Pointing Survey: Formation Emphasis



(a) 26 Points (b) 98 Points (c) 290 Points

Figure: Formation Emphasis: Comparison of 26, 98, 290 Points

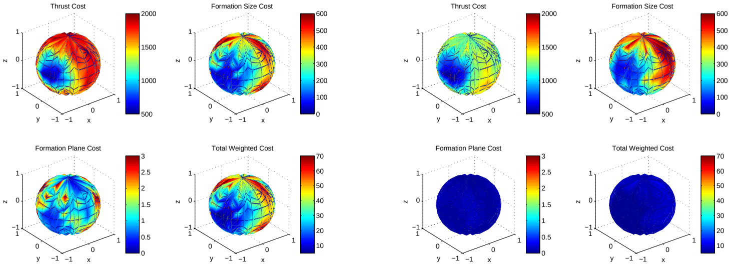
Pointing Survey: Plane Emphasis



(a) 26 Points (b) 98 Points (c) 290 Points

Figure: Plane Emphasis: Comparison of 26, 98, 290 Points

Formation vs. Plane Emphasis Comparison

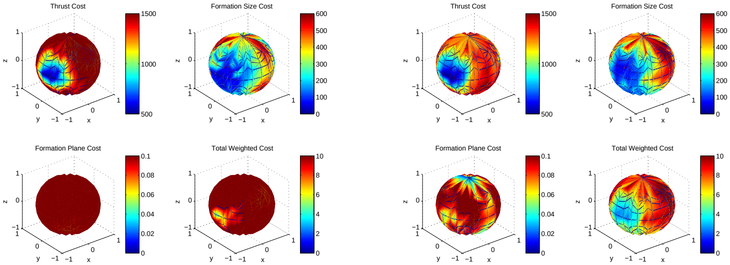


(a) Formation Emphasis

(b) Plane Emphasis

Figure: Formation vs. Plane Emphasis: Scaling for Formation Emphasis

Formation vs. Plane Emphasis Comparison



(a) Formation Emphasis

(b) Plane Emphasis

Figure: Formation vs. Plane Emphasis: Scaling for Plane Emphasis

Conclusions

- A modified collocation method with a segment-time switching algorithm leads to highly constrained control solutions
- Generalized formulation allows users to input
  - formation configuration, size, orientation, and rotation rate
  - thruster capability and placement
  - dynamic model and reference trajectory
  - initial and terminal conditions
- Suited to aid in establishing requirements and capabilities for highly constrained formations

Conclusions

- This investigation explores the range of applications of the FSCT method
  - The applicability of the method extends to all engineering disciplines
- FSCT vs. Multiple Lyapunov Functions
  - Optimal control laws may be extracted whose performance exceeds those derived using a Lyapunov argument
- Multiple independent decision inputs managed simultaneously
- Solutions derived via the FSCT method are utilized in conjunction with a hybrid system model predictive control scheme
  - Optimized control schedules can be realized in the context of potential perturbations or other unknowns
- Some continuous control input systems may be more accurately described as systems ultimately relying on discrete decision variables
  - Continuous control variables may often be extended into a set of continuous state variables and discrete inputs