

# SOLUTION

## ECE 538 Digital Signal Processing I Final Exam 2015 Test Date: Tuesday, 16 December 2015

### Cover Sheet

Test Duration: 120 minutes.

Open Book but Closed Notes.

Four 8.5 x 11 handwritten crib sheets allowed.

Calculators NOT allowed.

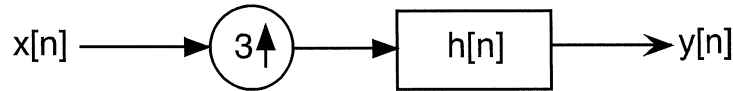
This test contains **FOUR** problems.

All work should be done in the proper space provided.

*Good luck in this final exam!  
Great having you in class this semester!  
Have a joyous holiday season!*

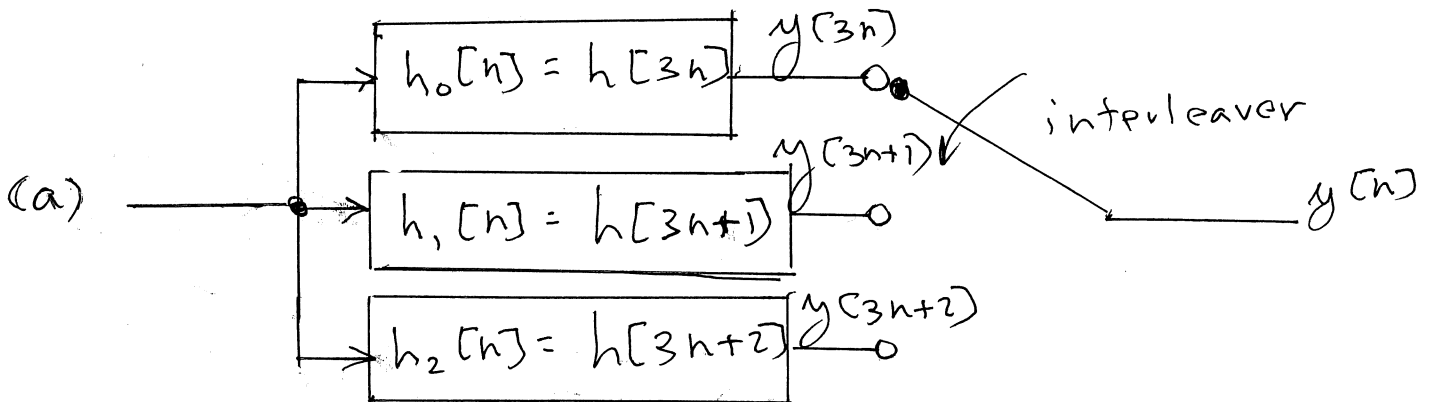
**Problem 1.** Consider the upsampler system below in Figure 1 and consider that the signal input to this system is

$$\text{Input Signal: } x[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n} \frac{\sin(\frac{\pi}{4}n)}{\pi n} \quad (1)$$



**Figure 1.**

- Draw block diagram of efficient implementation of the upsampler system in Figure 1.
- For the ideal case where  $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$ , consider the filter  $h_0[n] = h[3n]$  and determine an expression for the output of this filter, denoted  $y_0[n] = x[n] * h_0[n]$ , when the input is the signal defined in Eqn (1) above.
- For the ideal case where  $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$ , consider the filter  $h_1[n] = h[3n + 1]$  and determine an expression for the output of this filter, denoted  $y_1[n] = x[n] * h_1[n]$ , when the input is the signal defined in Eqn (1) above.
- For the ideal case where  $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$ , consider the filter  $h_2[n] = h[3n + 2]$  and determine an expression for the output of this filter, denoted  $y_2[n] = x[n] * h_2[n]$ , when the input is the signal defined in Eqn (1) above.
- For the ideal case where  $h[n] = 3 \frac{\sin(\frac{\pi}{3}n)}{\pi n}$ , determine an expression for the overall output of this system, denoted  $y[n]$ , when the input is the signal defined in Eqn (1) above.



This page left intentionally blank for student work for Problem 1.

$$(b) \quad h[3n] = f(n) \quad y_0[n] = x[n]$$

$$(c) \quad h_1[n] = h[3n+1] = \frac{\sin\left(\pi\left(n+\frac{1}{3}\right)\right)}{\pi\left(n+\frac{1}{3}\right)} \quad \neq h$$

$$y_1[n] = x\left[n+\frac{1}{3}\right] = \frac{\sin\left(\frac{3\pi}{4}\left(n+\frac{1}{3}\right)\right)}{\pi\left(n+\frac{1}{3}\right)} \sin\left(\frac{\pi}{4}\left(n+\frac{1}{3}\right)\right)$$

$$(d) \quad h_2[n] = h[3n+2] = \frac{\sin\left(\pi\left(n+\frac{2}{3}\right)\right)}{\pi\left(n+\frac{2}{3}\right)}$$

$$y_2[n] = x\left[n+\frac{2}{3}\right] = \frac{\sin\left(\frac{3\pi}{4}\left(n+\frac{2}{3}\right)\right)}{\pi\left(n+\frac{2}{3}\right)} \sin\left(\frac{\pi}{4}\left(n+\frac{2}{3}\right)\right)$$

$$(e) \quad y[n] = x\left[\frac{n}{3}\right] = \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \cdot \frac{\sin\left(\frac{\pi}{12}n\right)}{\pi n} \cdot 9$$

# Digital Signal Processing I

# Final Exam 2015

**Problem 2.** A second-order digital filter is to be designed from an analog filter having two poles in the s-plane at  $p_1 = -1 + 2j$  and  $p_2 = -1 - 2j$  and two zeros at  $z_1 = j$  and  $z_2 = -j$ ,

$$H_a(s) = G \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

via the bilinear transformation method characterized by the mapping

$$s = \frac{z - 1}{z + 1}$$

- (a) Is the resulting digital filter (BIBO) stable? Briefly explain why or why not.
- (b) Denote the frequency response of the resulting digital filter as  $H(\omega)$  (the DTFT of its impulse response). You are given that in the range  $0 < \omega < \pi$ , there is only one value of  $\omega$  for which  $H(\omega) = 0$ . Determine that value of  $\omega$ .
- (c) Draw a pole-zero diagram for the resulting **digital** filter. Give the exact locations of the poles and zeros of the digital filter in the z-plane.
- (d) Plot the magnitude of the DTFT of the resulting digital filter,  $|H(\omega)|$ , over  $-\pi < \omega < \pi$ . You are given that  $H(0) = 0.8$ . Be sure to indicate any frequency for which  $|H(\omega)| = 0$ . Also, specifically note the numerical value of  $|H(\omega)|$  for  $\omega = \frac{\pi}{2}$  and  $\omega = \pi$ .
- (e) Determine the difference equation for the resulting digital filter.

(a) Since analog poles are in LHP  $\Rightarrow$  bilinear transform guarantees they will be mapped to poles inside unit circle

$$z = \frac{1+s}{1-s} \quad \text{pole 1: } \frac{1+p_1}{1-p_1} = \frac{1+(-1+2j)}{1-(-1+2j)} = -\frac{1}{2} + j\frac{1}{2}$$

$$\text{pole 2} = -\frac{1}{2} - j\frac{1}{2}$$

$$\text{zeros: } \text{zero 1: } \frac{1+z_1}{1-z_1} = \frac{1+j}{1-j} = \frac{\sqrt{2} e^{j\frac{\pi}{4}}}{\sqrt{2} e^{-j\pi/4}} = e^{j\frac{\pi}{2}} = j$$

$$\text{zero 2: } -j$$

This page left intentionally blank for student work for Problem 2.

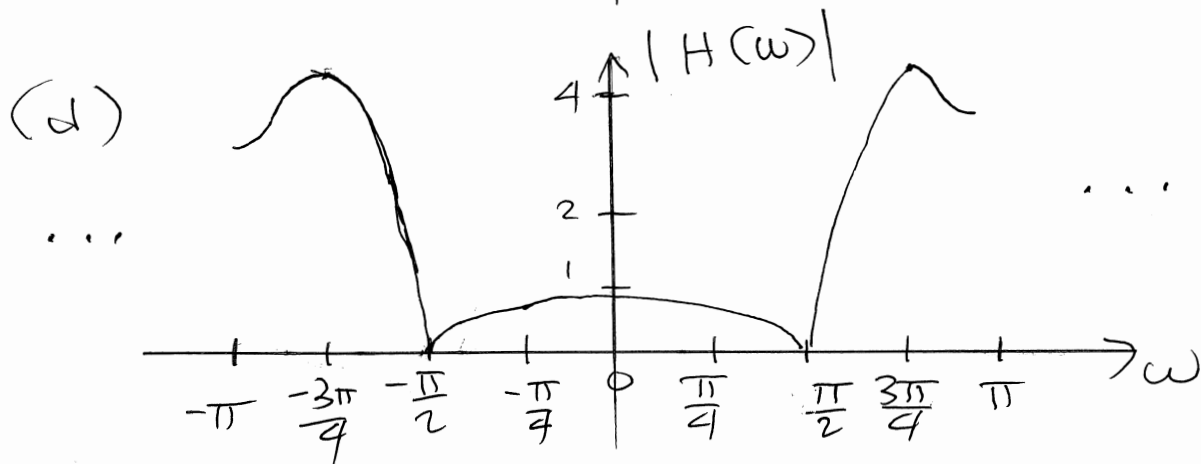
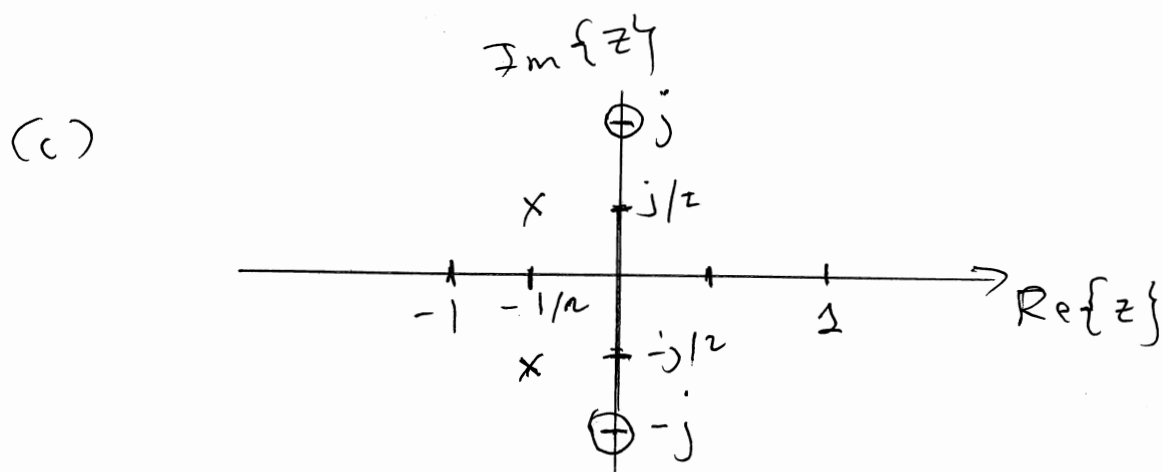
(a) Note: easy to show that complex-conjugate pair of poles (zeros) in analog domain mapped to complex-conjugate pair of poles (zeros) in digital domain

(b) Since zero on unit circle at  $\omega = \pm \frac{\pi}{2}$   
and poles at  $-\frac{3\pi}{4}, +\frac{3\pi}{4}$

For  $0 < \omega < \pi$   $H(\omega) = 0$  at  $\omega = \frac{\pi}{2}$

$$H\left(\frac{\pi}{2}\right) = 0$$

$$H\left(-\frac{\pi}{2}\right) = 0$$



This page left intentionally blank for student work for Problem 2.

$$(e) \quad H(z) = G \frac{(z+j)(z-j)}{(z+\frac{1}{2}-j\frac{1}{2})(z+\frac{1}{2}+j\frac{1}{2})}$$

$$= G \frac{z^2 + 1}{(z+\frac{1}{2})^2 + \frac{1}{4}} = G \frac{z^2 + 1}{z^2 + z + \frac{1}{2}}$$

$$\text{at } z=1 \quad (\omega=0) : \quad G \frac{1+1}{1+1+\frac{1}{2}} = .8$$

$$G \frac{2}{\frac{5}{2}} = \frac{4}{5} \Rightarrow G = 1$$

$$y[n] + y[n-1] + \frac{1}{2}y[n-2] = x[n] + x[n-2]$$

**Problem 3.** Consider the length-7 DT signal below where the first value corresponds to  $n = 0$ .

$$x[n] = \{+1, +1, +1, -1, -1, +1, -1\}$$

One way to compute autocorrelation is  $r_{xx}[n] = x[n] * x^*[-n]$ . Suppose we want to use DFT-based processing to compute autocorrelation. One issue is that the DFT assumes that signals start at  $n = 0$ . So, let's time-shift the time-reversed signal to the right by  $7-1=6$  to form  $x[-(n-6)]$  so that it starts at  $n = 0$ ;  $h[n] = x[-(n-6)]$  is the causal matched filter. Since  $x[n]$  is real-valued,

$$h[n] = x[-(n-6)] \xleftrightarrow{DTFT} H(\omega) = X^*(\omega) e^{-j6\omega}$$

- (a) A 16-point DFT of  $x[n]$  is computed and denoted as  $X_{16}(k)$ ,  $k = 0, 1, \dots, 15$ . We then form  $Y_{16}(k)$  via the point-wise product defined below

$$Y_{16}(k) = X_{16}(k) X_{16}^*(k) e^{-j6 \frac{k2\pi}{16}} \quad k = 0, 1, \dots, 15$$

Finally, compute  $y_{16}[n]$  as the 16-point Inverse DFT of  $Y_{16}(k)$ . Determine and list all 16 values of  $y_{16}[n]$ , for  $n = 0, 1, \dots, 15$ . Show all work.

- (b) A 13-point DFT of  $x[n]$  is computed and denoted as  $X_{13}(k)$ ,  $k = 0, 1, \dots, 12$ . We then form  $Y_{13}(k)$  via the point-wise product defined below

$$Y_{13}(k) = X_{13}(k) X_{13}^*(k) e^{-j6 \frac{k2\pi}{13}} \quad k = 0, 1, \dots, 12$$

Finally, compute  $y_{13}[n]$  as the 13-point Inverse DFT of  $Y_{13}(k)$ . Determine and list all 13 values of  $y_{13}[n]$ , for  $n = 0, 1, \dots, 12$ .

- (c) An 10-point DFT of  $x[n]$  is computed and denoted as  $X_{10}(k)$ ,  $k = 0, 1, \dots, 9$ . We then form  $Y_{10}(k)$  via the point-wise product defined below

$$Y_{10}(k) = X_{10}(k) X_{10}^*(k) e^{-j6 \frac{k2\pi}{10}} \quad k = 0, 1, \dots, 9$$

Finally, compute  $y_{10}[n]$  as the 10-point Inverse DFT of  $Y_{10}(k)$ . Determine and list all 10 values of  $y_{10}[n]$ , for  $n = 0, 1, \dots, 9$ .

- (d) An 16-point DFT of  $x[n]$  is computed and denoted as  $X_{16}(k)$ ,  $k = 0, 1, \dots, 15$ . We then form  $Y_{16}(k)$  via the point-wise product defined below

$$Y_{16}(k) = X_{16}(k) X_{16}^*(k) e^{-j6 \frac{k2\pi}{16}} G_{16}(k) \quad k = 0, 1, \dots, 15$$

where  $G_{16}(k)$  is the 16-point DFT of the 2-tap filter  $g[n] = \{1, 1\}$ . Finally, compute  $x_{16}[n]$  as the 16-point Inverse DFT of  $Y_{16}(k)$ . Determine and list all 16 values of  $x_{16}[n]$ , for  $n = 0, 1, \dots, 15$ .

Consider:  $y[n] = x[n] * x^*[-n] * \delta[n-6]$

nonzero over  
 $-6 \leq n \leq 6$

$r_{xx}[n]$   
 $= r_{xx}[n-6]$

} auto correlation  
 shifted to right  
 by 6 to start  
 at  $n=0$

length = 13

(a) since  $16 > 13$ :

$y_{16}[n] = \{-1, 0, -1, 0, -1, 0, 7, 0, -1, 0, -1, 0, -1, 0, 0, 0\}$   
 auto correlation of Barker code  
 of length 7

(b)  
 $y_{13}[n] = \{-1, 0, -1, 0, -1, 0, 7, 0, -1, 0, -1\}$

(c) Since  $10 < 13$  and  $13-10 = 3$  points at end  
 of  $y_{13}[n]$  aliased into 3 points at beginning

$y_{10}[n] = \{-2, 0, -2, 0, -1, 0, 7, 0, -1, 0\}$

(d) Consider  $z[n] = x[n] * x^*[-n] * \delta[n-6] * \{1, 1\}$   
 of length  $13 + 2 - 1 = 14$

Since  $16 > 14$  and

$\{-1, 0, -1, 0, -1, 0, 7, 0, -1, 0, -1\}$

Ans,  $+ \{-1, 0, -1, 0, -1, 0, 7, 0, -1, 0, -1\}$

$y_{16}[n] = \{-1, -1, -1, -1, -1, -1, 7, 7, -1, -1, -1, -1, -1, -1, 0, 0\}$   
 $16 - 14 = 2$



## Problem 4.

- (a) Consider sampling the DTFT below at  $N = 8$  equi-spaced frequencies over  $0 \leq \omega < 2\pi$ .

$$X(\omega) = \frac{1}{1 - 0.8e^{-j\omega}} + \frac{1}{1 - 0.9e^{-j\omega}}$$

That is, let  $X_8(k) = X(2\pi k/8)$ ,  $k = 0, 1, \dots, 7$ , where  $X(\omega)$  is defined above. Now, compute  $x_8[n]$  as the 8-point Inverse DFT of  $X_8(k)$ . Determine a closed-form ("plug-and-chug") expression for  $x_8[n]$  that prescribes the value of  $x_8[n]$  for  $n = 0, 1, \dots, 7$ . Show all work. **Note:** A *closed-form* expression contains NO summations and it is NOT a listing of numbers. *Hint:*

$$C_1 = \frac{1}{1 - (.9)^8} = 1.7558 \quad C_2 = \frac{1}{1 - (.8)^8} = 1.2016$$

- (b) Show how your answer changes when everything is the same as defined in part (a) except that the DTFT is now defined as below.

$$X(\omega) = \frac{8}{1 - 0.8e^{-j\omega}} - \frac{4}{1 - 0.9e^{-j\omega}}$$

- (c) Show how your answer changes when everything is the same as defined in part (a) except that the DTFT is now defined as below.

$$X(\omega) = \frac{e^{j3\omega}}{1 - 0.8e^{-j\omega}} + \frac{e^{j5\omega}}{1 - 0.9e^{-j\omega}}$$

- (d) Show how your answer changes when everything is the same as defined in part (a) except that the DTFT is now defined below. (That is, the two terms are now multiplied rather than being summed as in part (a).) Show all work.

$$X(\omega) = \frac{1}{1 - 0.8e^{-j\omega}} \frac{1}{1 - 0.9e^{-j\omega}}$$

$C_1, C_2$  defined above

Note: under week 9, see notes entitled:

"Time-Domain Aliasing of Multi-Pole Causal Signals"

$$(a) \quad x_8[n] = \{C_1 (.8)^n + C_2 (.9)^n\} \{u[n] - u[n-8]\}$$

(b) linearity holds:

$$X_8[n] = \{8c_1 (.8)^n - 4c_2 (.9)^n\} \{u[n] - u[n-8]\}$$

(c) yields circulant shift<sub>S</sub>:

$$X_8[n] = \sum_{l=0}^{\infty} (.8)^{n+3-2l} \{u[n] - u[n-8]\}$$

circulant  
shift  
"left"  
by 3

$$+ \sum_{l=0}^{\infty} (.9)^{n+5-2l} \{u[n] - u[n-8]\}$$

circulant  
shift to  
"left"  
by 5

In this case:

$$X_8[n] = c_1 \{ .8^3, .8^4, .8^5, .8^6, .8^7, .8^0, .8^1, .8^2 \} \\ + c_2 \{ .9^5, .9^6, .9^7, .9^0, .9^1, .9^2, .9^3, .9^4 \}$$

$$(d) X_8[n] = c_1 (.8)^n \{u[n] - u[n-8]\} \oplus c_2 (.9)^n \{u[n] - u[n-8]\}$$

$$= c_1 c_2 \cdot \{ .8^0, .8^1, .8^2, .8^3, .8^4, .8^5, .8^6, .8^7 \} \oplus \{ .9^0, .9^1, .9^2, .9^3, .9^4, .9^5, .9^6, .9^7 \}$$

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$$x_8[n] = \sum_{l=-\infty}^{\infty} y[n+8l] \{u[n] - u[n-8]\}$$

$$y[n] = .8^n u[n] * .9^n u[n]$$
$$= \left\{ \sum_{k=0}^n .8^k .9^{n-k} \right\} u[n]$$

$$= (.9)^n \sum_{k=0}^n \left( \frac{.8}{.9} \right)^k$$
$$= \left\{ (.9)^n \frac{1 - \left( \frac{.8}{.9} \right)^{n+1}}{1 - \frac{.8}{.9}} \right\} u[n]$$

$$x_8[n] = \sum_{l=0}^{\infty} y[n+8l] \{u[n] - u[n-8]\}$$

only shifts to left  
fall in  $0 \leq n \leq 7$

note:

$$\alpha^n u[n] * \beta^n u[n]$$

$$= \left( \frac{\alpha}{\alpha - \beta} \right) \alpha^n u[n] + \left( \frac{\beta}{\beta - \alpha} \right) \beta^n u[n]$$

$$\text{use } \alpha = .8 > \beta = .9$$