SOLUTION

NAME: **30 Oct. 2015**

ECE 538 Digital Signal Processing I Exam 2 Fall 2015

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET
Test Duration: 60 minutes.
Open Book but Closed Notes.
One (both sides) handwritten 8.5 in x 11 in crib sheet allowed
Calculators NOT allowed.
All work should be done in the space provided.

There are two problems.
Problem 1 has 4 parts, 1(a) thru 1(d).
Problem 2 has 6 parts, 2(a) thru 2(f).

Continuous-Time Fourier Transform (Hz): $X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$

Continuous-Time Fourier Transform Pair (Hz): $\mathcal{F}\left\{\frac{\sin(2\pi Wt)}{\pi t}\right\} = rect\left\{\frac{F}{2W}\right\}$ where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = X_1(F) * X_2(F)$, where * denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(F)$, $i = 1, 2$.

Relationship between DTFT and CTFT frequency variables in Hz: $\omega = 2\pi \frac{F}{F_s}$, where $F_s = \frac{1}{T_s}$ is the sampling rate in Hz

Prob. 1(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ with $F_s = \frac{1}{T_s} = 8W$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work. **NOTE:** The signal $x_0[n]$ is the input signal for each of the three remaining parts of this problem.

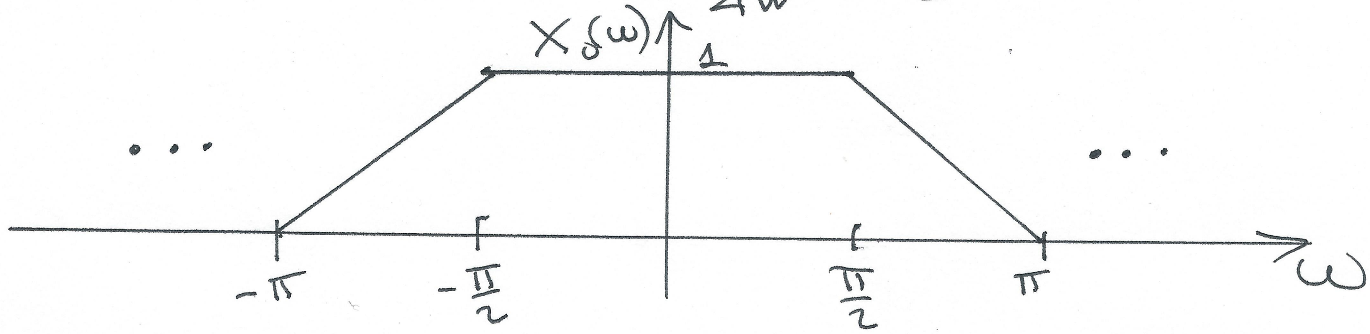
$$x_0(t) = T_s \frac{1}{2W} \frac{\sin(2\pi W t)}{\pi t} \frac{\sin(2\pi 3W t)}{\pi t}$$

- max frequency $W + 3W$ is mapped to:

$$\omega = 2\pi \frac{4W}{8W} = \pi$$

- end of flat part $3W - W = 2W$ mapped to:

$$\omega = 2\pi \frac{2W}{4W} = \frac{\pi}{2}$$



$F_s^{(0)} = 8W \quad T_s^{(0)} = \frac{1}{8W}$

1(b) The discrete-time signal, $x_1[n]$, is created by running $x_0[n]$ from part (a) thru the DT system below. You don't have to do a lot of work but clearly explain your answers.

$$h_1[n] = 3\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$$

$$x_0[n] \longrightarrow \boxed{3\uparrow} \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{2\downarrow} \longrightarrow x_1[n]$$

(i) Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$.

(ii) What is the new effective sampling rate, $F_s^{(1)}$, at the output relative to original sampling rate $F_s = 8W$?

- no aliasing, so $F_{s_{new}}^{(1)} = \frac{3}{2} F_s = \frac{3}{2} \cdot 8W = 12\,W$

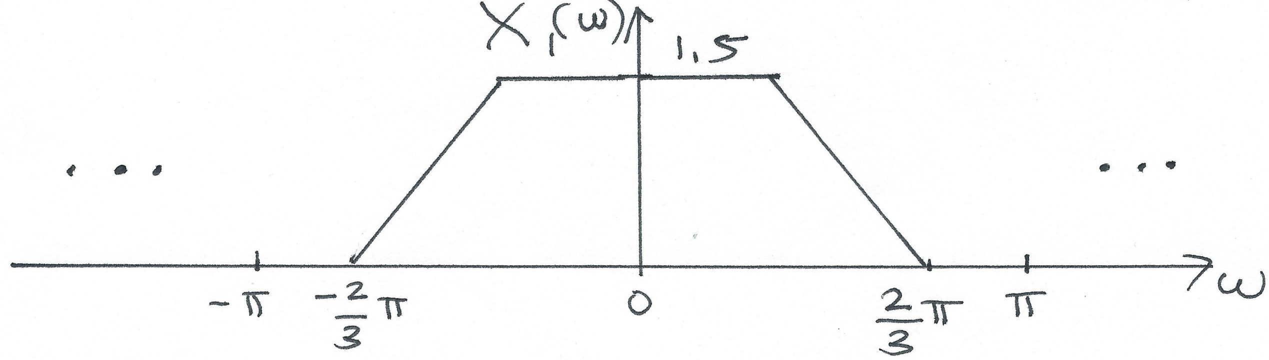
$$T_s^{(1)} = \frac{1}{12W}$$

- new max freq. in DT frequency domain:

$$\omega_M = \frac{2}{3}\pi = 2\pi \frac{4W}{12W} = \frac{2}{3}\pi$$

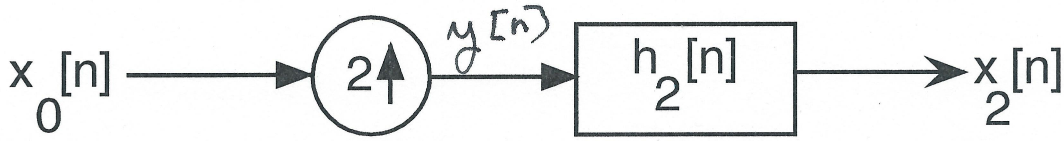
- The decimation by 2 causes a gain reduction by 2 (in amplitude)

combined with filter gain of 3 $\Rightarrow$ new height $= \frac{3}{2} = 1.5$



1(c) The discrete-time signal, $x_2[n]$, is created by running $x_0[n]$ from part (a) through the DT system below.

$$h_2[n] = 2\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

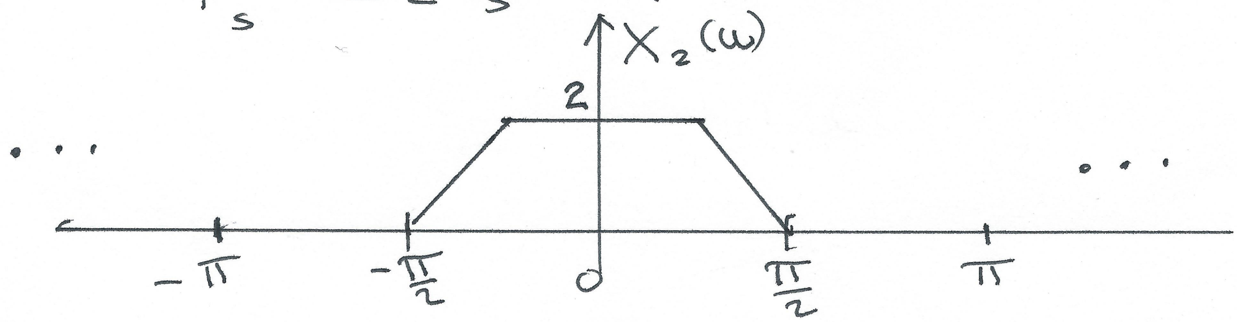


(i) Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$.

(ii) Express the new effective sampling rate, $F_s^{(2)}$, at the output in terms of the original sampling rate $F_s = 8W$.

• This is upsampling by a factor of 2

$F_s^{(2)} = 2F_s = 2(8W) = 16W \qquad T_s^{(2)} = \frac{1}{16W}$



$X_2(\omega) = X_0(2\omega)\,H_2(\omega)$

= plot above

1(d) The discrete-time signal, $x_3[n]$, is created by running $x_0[n]$ from part (a) through the DT system below; "the output" refers to $x_3[n]$ in all parts below.

$$x_0[n] \longrightarrow \boxed{2\uparrow} \longrightarrow \boxed{h_3[n]} \longrightarrow \boxed{3\downarrow} \longrightarrow x_3[n]$$

$$h_3[n] = 3\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$$

(i) Plot the magnitude of the DTFT of $x_3[n]$, $|X_3(\omega)|$, over $-\pi < \omega < \pi$.

(ii) Express the new effective sampling rate, $F_s^{(3)}$, at the output in terms of the original sampling rate $F_s = 8W$?

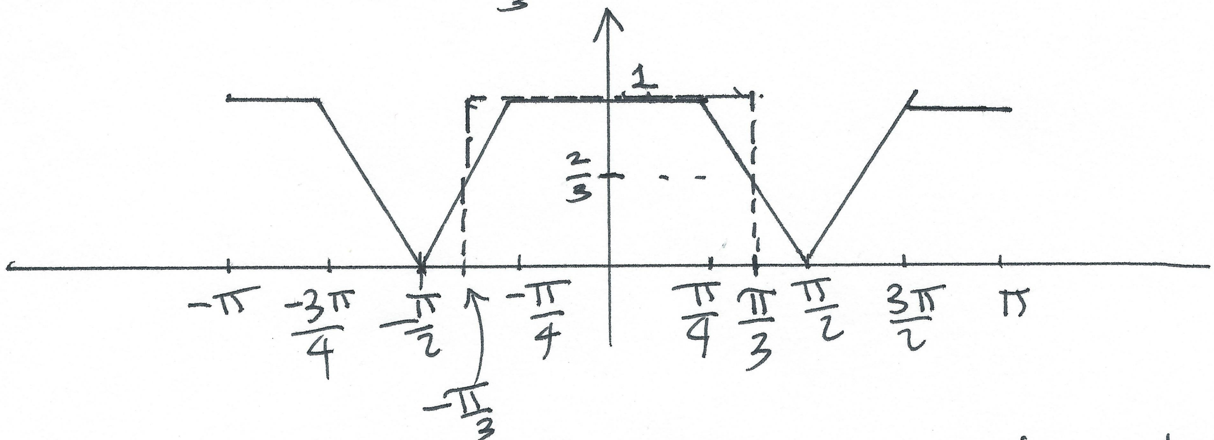
(iii) Is there aliasing in the output? Yes or No. Briefly explain.

(iv) Is there loss of high frequency content in the output? Yes or No. Briefly explain.

(v) Express the output $x_3[n]$ in terms of a sampled version of the lowpass-filtered signal $x_{LP}(t)$ below, where $x_0(t)$ was defined in part (a). You can use $x_{LP}(t)$ in your answer to Problem 2(b.)

$$x_{LP}(t) = x_0(t) * \frac{\sin\left(2\pi\frac{8W}{3}t\right)}{\pi t}$$

- this effects a new sampling rate of $F_s^{(3)} = \frac{2}{3} F_s$
- since $F_s = 8W$ is Nyquist rate, this effects a sampling rate below the Nyquist rate
- But the filter $h_3[n]$ removes high frequency content above $\frac{\pi}{3}$ $\Rightarrow$ thus, no aliasing



- Filter gain of 3 offset by gain reduction by 3 due to decimation by 3

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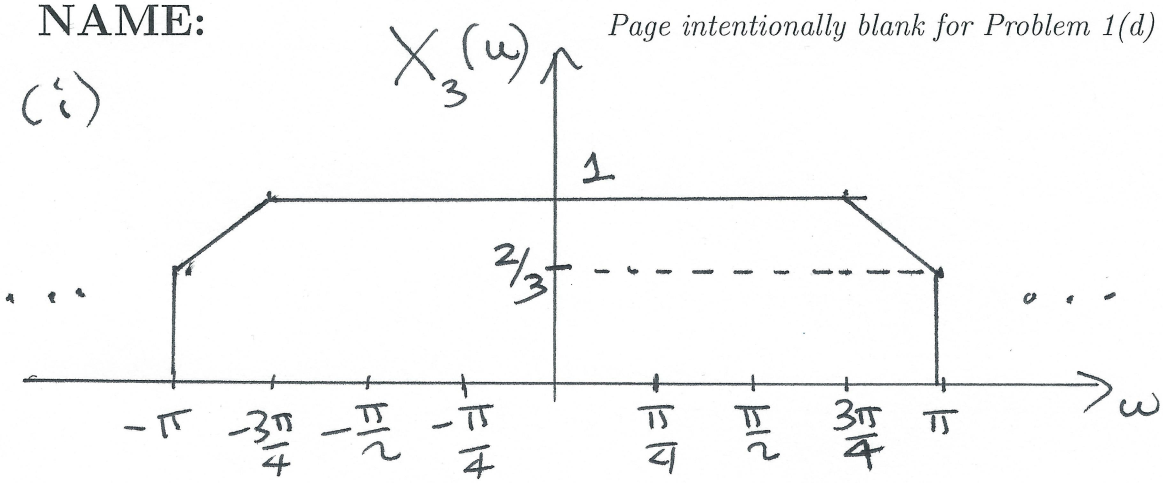
(i)

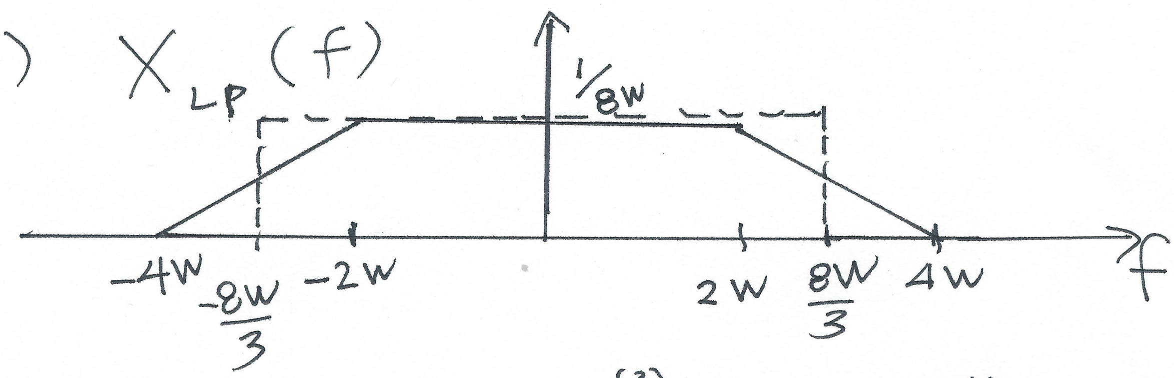


(ii) $F_s^{(3)} = \frac{2}{3} F_s = \frac{2}{3} \cdot 8W = \frac{16W}{3}$ $\qquad T_s^{(3)} = \frac{3}{16W}$

(iii) No aliasing

(iv) Yes, loss of high frequency content $\Rightarrow$ removed in order to avoid aliasing

(v)



new sampling rate : $F_s^{(3)} = \frac{2}{3} 8W = \frac{16W}{3}$

$2W$ is mapped to $2\pi \frac{2W}{16W/3} = \frac{3\pi}{4}$

$\frac{8W}{3}$ is mapped to $2\pi \frac{\frac{8}{3}W}{\frac{16}{3}W} = \pi$

$\Bigg\}$ Same as $X_3(\omega)$ above

$\frac{1}{8W} \cdot \alpha = \frac{1}{\frac{16W}{3}} \Rightarrow \frac{3}{2}$

$x_3[n] = \frac{3}{2} x_{LP}(t) \Big|_{t = n\frac{3}{16W}}$

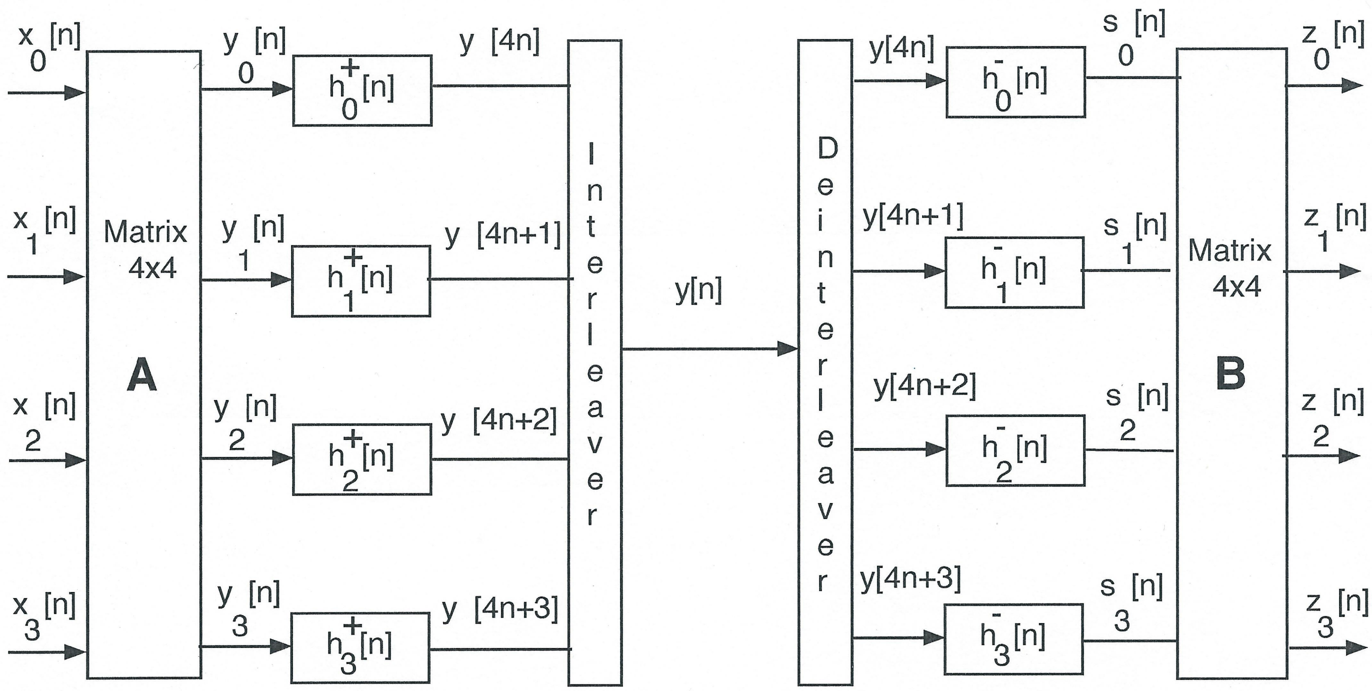


Figure 1.

Problem 2. This problem is about digital subbanding of the four DT signals $x_i[n]$, $i = 0, 1, 2, 3$ from Problem 1. Digital subbanding of these four signals is effected in the efficient way via filter bank in Figure 1. All of the quantities in Figure 1 are defined below: the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

$$h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 1 are defined as

$$h_\ell^+[n] = h_{LP}[4n+\ell], \quad \ell = 0, 1, 2, 3. \tag{2}$$

The respective signals at the inputs to these filters are formed from the input signals as

$$\begin{bmatrix} y_0[n] \\ y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-j\frac{2\pi}{4}} & 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} \\ e^{-j\frac{2\pi(2)}{4}} & 1 & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{4\pi(2)}{4}} \\ e^{-j\frac{2\pi(3)}{4}} & 1 & e^{j\frac{2\pi(3)}{4}} & e^{j\frac{4\pi(3)}{4}} \end{bmatrix} \begin{bmatrix} x_0[n] \\ x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-j\frac{2\pi}{4}} & 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} \\ e^{-j\frac{2\pi(2)}{4}} & 1 & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{4\pi(2)}{4}} \\ e^{-j\frac{2\pi(3)}{4}} & 1 & e^{j\frac{2\pi(3)}{4}} & e^{j\frac{4\pi(3)}{4}} \end{bmatrix} \tag{3}$$

The polyphase component filters on the right side of Figure 1 are defined as

$$h_\ell^-[n] = h_{LP}[4n-\ell], \quad \ell = 0, 1, 2, 3. \tag{4}$$

The final output signals (on the far right side of Figure 1) are formed from linear combinations of the outputs of these filters via the matrix transformation below.

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{2\pi(3)}{4}} \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi(2)}{4}} & e^{-j\frac{2\pi(3)}{4}} \\ 1 & e^{-j\frac{4\pi}{4}} & e^{-j\frac{4\pi(2)}{4}} & e^{-j\frac{4\pi(3)}{4}} \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{2\pi(3)}{4}} \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi(2)}{4}} & e^{-j\frac{2\pi(3)}{4}} \\ 1 & e^{-j\frac{4\pi}{4}} & e^{-j\frac{4\pi(2)}{4}} & e^{-j\frac{4\pi(3)}{4}} \end{bmatrix} \tag{5}$$

$x_0[n] \Leftrightarrow X_0(\omega)$ centered at $\omega = -\frac{\pi}{2}$ — $X_1(\omega)$ at $\omega = 0$ — $X_2(\omega)$ at $\omega = \pi/2$ — $X_3(\omega)$ at $\omega = \pi$

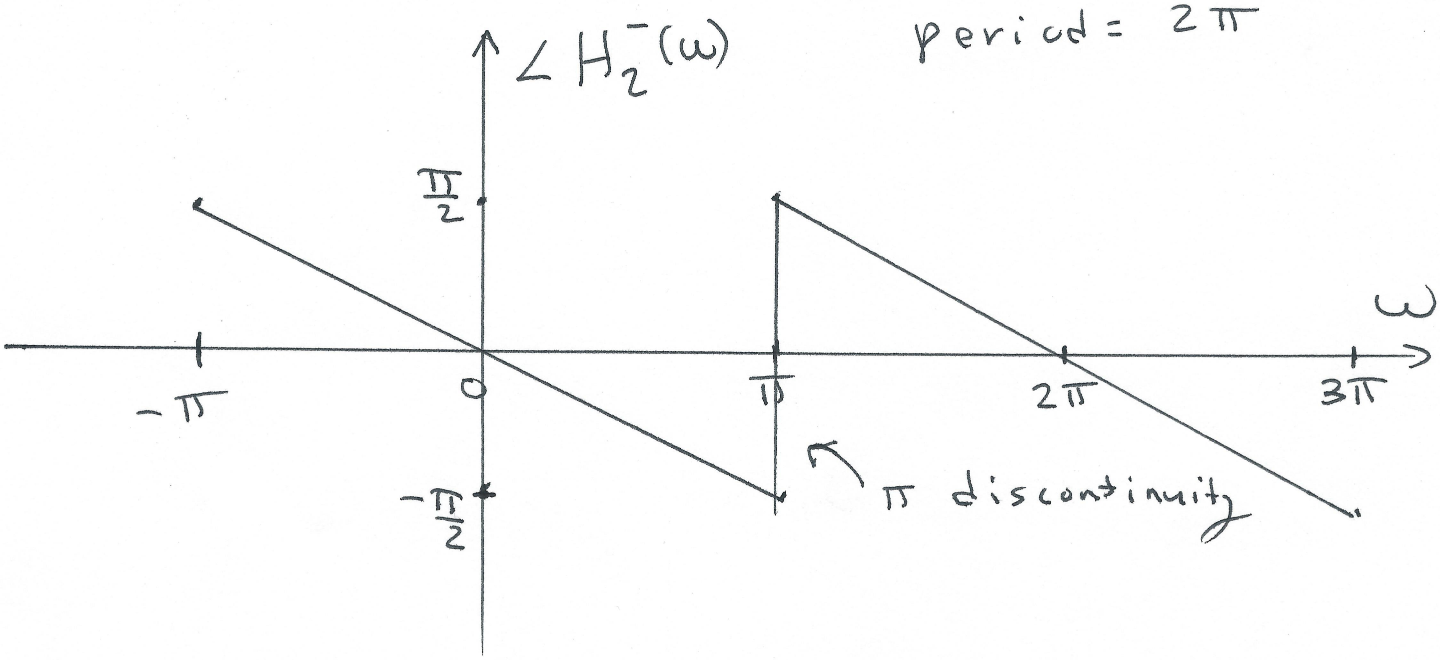
Problem 2, part (a). Show all work. For all parts of this problem, $h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$.

(a) (i) Write a simple expression for the DTFT, $H_2^-(\omega)$, of $h_2^-[n] = h_{LP}[4n-2]$ that holds for $-\pi < \omega < \pi$.

(ii) Plot the phase $\angle H_2^-(\omega)$ over $-\pi < \omega < 3\pi$. Note that I am asking you to plot the phase, $\angle H_2^-(\omega)$, from $-\pi$ to 3π.

$H_2^-(\omega) = e^{-j\frac{2}{4}\omega} = e^{-j\frac{\omega}{2}}$ over $-\pi < \omega < \pi$

period $= 2\pi$



2(b) Express the output of the filter $h_2^+[n] = h_{LP}[4n+2]$ in terms of sampled and time-shifted versions of the original analog input signal $x_0(t)$ and also $x_{LP}(t)$ defined in Prob. 1(d). You don't need to write out the expressions for $x_0(t)$ and $x_{LP}(t)$. Keep in mind that all of the signals correspond to different sampling rates; make sure that's clear in your answer. You don't have to do a lot of work here but explain answers.

3rd row of A is: $[-1 \quad 1 \quad -1 \quad 1]$

$$y[4n+2] = -x_0\left(nT_s^{(0)} + \frac{T_s^{(0)}}{2}\right) + x_0\left(nT_s^{(1)} + \frac{T_s^{(1)}}{2}\right)$$

$$- x_0\left(nT_s^{(2)} + \frac{T_s^{(2)}}{2}\right) + \frac{3}{2} x_{LP}\left(nT_s^{(3)} + \frac{T_s^{(3)}}{2}\right)$$

$T_s^{(0)} = \frac{1}{8W}$ $\quad T_s^{(1)} = \frac{1}{12W}$ $\quad T_s^{(2)} = \frac{1}{16W}$ $\quad T_s^{(3)} = \frac{3}{16W}$

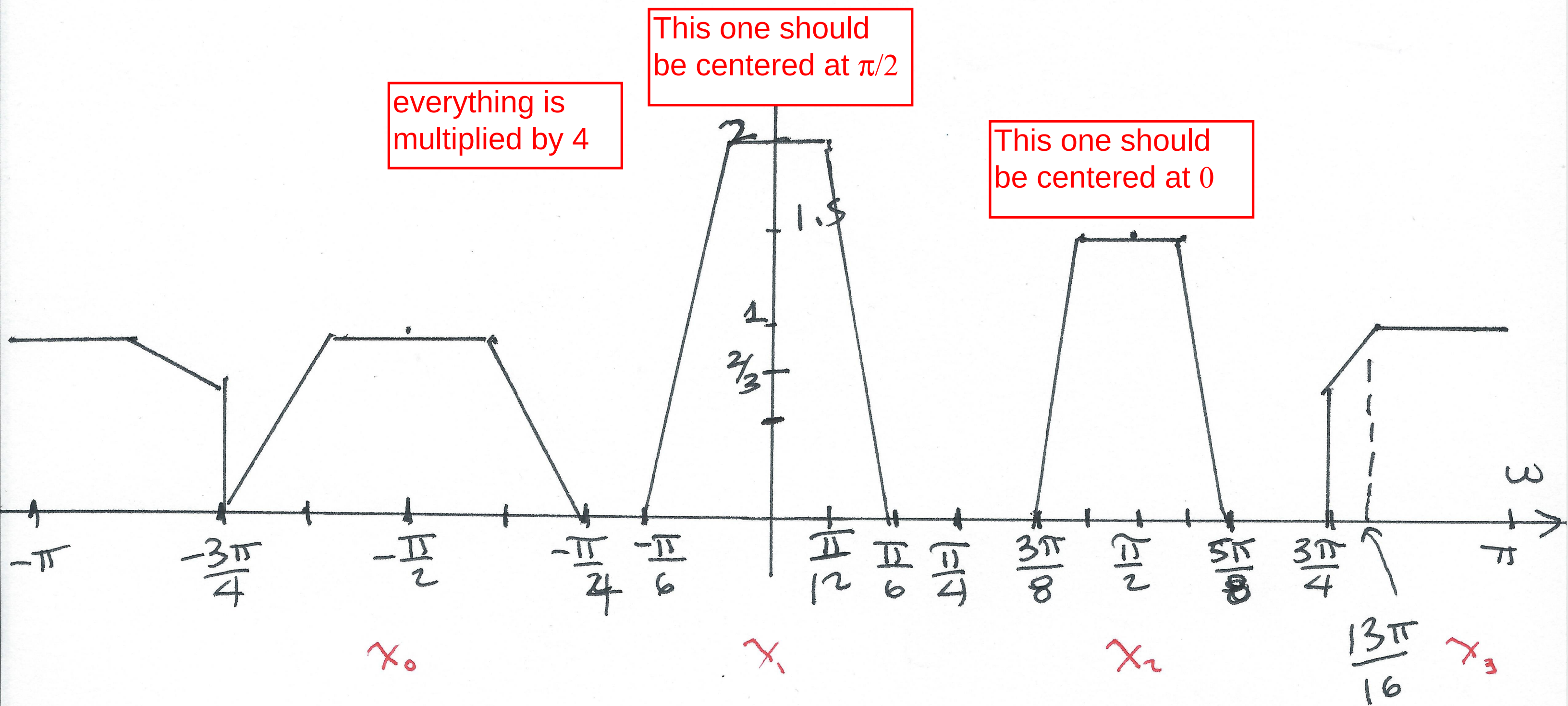
2(c) The output of the filter $h_3^-[n] = h_{LP}[4n-3]$ is denoted $s_3[n]$ in the block diagram. Express $s_3[n]$ in terms of $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$. You don't need to write out the expressions for $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$. You don't have to do a lot of work here; briefly explain your answer.

Since $h_3^+[n] * h_3^-[n] = \delta[n]$, $s_3[n] = y_3[n]$

last row of $A = [\,j \;\; 1 \;\; -j \;\; -1\,]$

$s_3[n] = y_3[n] = j\,x_0[n] + x_1[n] - j\,x_2[n] - x_3[n]$

2(d) Plot the magnitude of the DTFT, $Y(\omega)$, of the interleaved signal $y[n]$ over $\pi < \omega < \pi$. Carefully label and graph the plot, clearly demarcating the subbands and showing which signal is in each subband. Clearly indicate the regions where $Y(\omega) = 0$.



2(e) Determine the convolution of $h_1^+[n] = h_{LP}[4n+1]$ with $h_3^+[n] = h_{LP}[4n+3]$, where $h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$. Simplify your answer as much as possible.

$$g[n] = h_1^+[n] * h_3^+[n] = h_{LP}[4n+1] * h_{LP}[4n+3] = ?$$

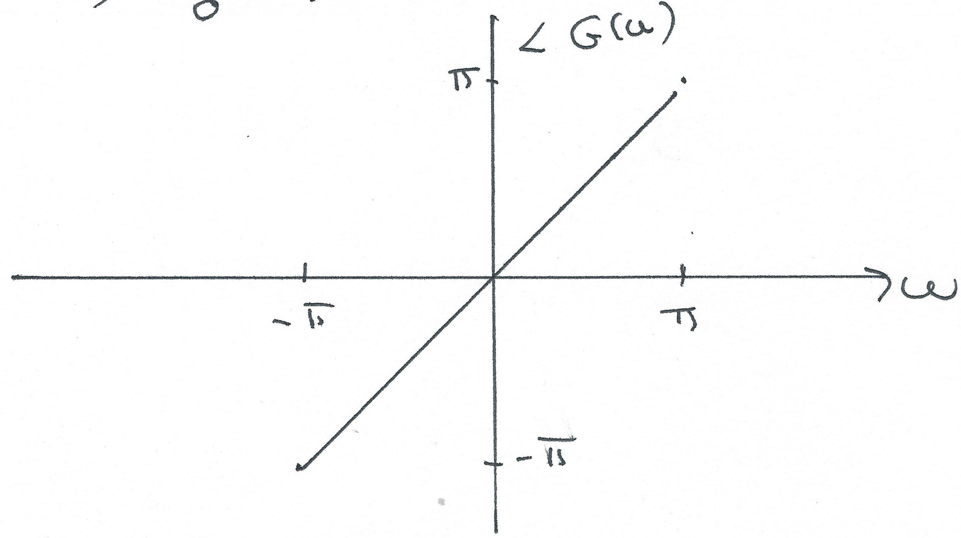
Plot the phase, $\angle G(\omega)$, of the DTFT of $g[n] = h_1^+[n] * h_3^+[n]$. This problem is most easily solved via frequency domain analysis. You must show and explain your work.

Over $-\pi < \omega < \pi$:

$$G(\omega) = H_1^+(\omega)\, H_3^+(\omega) = e^{j\frac{\omega}{4}} e^{j\frac{3\omega}{4}} =$$

$$= e^{j\omega}$$

$$\Rightarrow g[n] = \delta[n+1]$$



2(f) It is easy to show that $\mathbf{AB} = 4\mathbf{I}$ and $\mathbf{BA} = 4\mathbf{I}$, where $\mathbf{I}$ is the 4x4 identity Matrix. Plot the magnitude, $|Z_3(\omega)|$, of the DTFT of the output $z_3[n]$, over $-\pi < \omega < \pi$.

$z_3[n] = 4\, x_3[n]$