SOLUTION

EE538 Digital Signal Processing I — Fall 2011
Exam 1 — Monday, Oct. 3, 2011

Cover Sheet

Test Duration: 60 minutes.
Coverage: Chapters 1-5.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **two** problems.
Show your work in the space provided for each problem.
You must show all work for each problem to receive full credit.
Always simplify your answers as much as possible.

Prob. No.	**Topic(s)**	**Points**
1.	DT Autocorrelation, Cross-Correlation Correlation in terms of Convolution	60
2.	LTI Systems: Properties, Transfer Functions, Frequency Response	40

Problem 1. [60 points]

(a) Consider $x[n]$ to be a real-valued sequence with autocorrelation $r_{xx}[\ell]$. Express the autocorrelation sequence $r_{yy}[\ell]$ for $y[n] = x[-n]$ in terms of $r_{xx}[\ell]$. Be sure to show clearly how you arrived at your answer.

For real-valued sequence:

$$r_{xx}[\ell] = x[\ell] * x[-\ell]$$

$$y[n] = x[-\ell]$$

$$r_{yy}[\ell] = y[\ell] * y[-\ell]$$
$$= x[-\ell] * x[\ell]$$
$$= x[\ell] * x[-\ell]$$

since convolution is commutative

$$= r_{xx}[\ell]$$

(b) Determine and plot the autocorrelation sequence $r_{xx}[\ell]$ for $x[n]$ defined below with $p = \frac{1}{2}$. *Note:* This sequence is used in parts (c) thru (f).

$$x[n] = \frac{1}{p}\left\{\delta[n] + (p^2 - 1)p^n u[n]\right\} \qquad (1)$$

→ impulse response of all-pass filter

Substituting $p = \frac{1}{2}$:

$$x[n] = 2\,\delta[n] + \frac{-3/4}{1/2}\left(\frac{1}{2}\right)^n u[n]$$

$$= 2\,\delta[n] - \frac{3}{2}\,p^n u[n]$$

$$r_{xx}[\ell] = \left(2\,\delta[\ell] - \frac{3}{2}\,p^{\ell} u[\ell]\right) * \left(2\,\underbrace{\delta[-\ell]}_{=\delta[\ell]} - \frac{3}{2}\,p^{-\ell} u[-\ell]\right)$$

$$= 4\,\delta[\ell] - 3\,p^{\ell} u[\ell] - 3\,p^{-\ell} u[-\ell] + \frac{9}{4}\,\underbrace{\frac{1}{1-p^2}\,p^{|\ell|}}_{\text{Eg. 2.65 in text}}$$

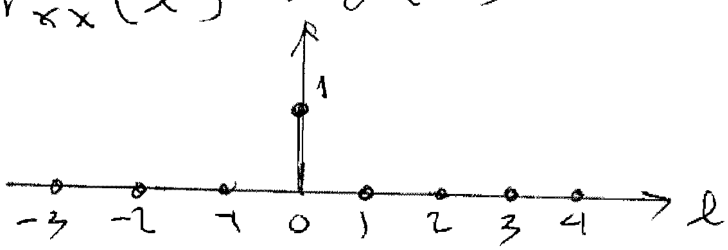
$$\frac{1}{1-p^2} = \frac{1}{3/4} = \frac{4}{3} \qquad \frac{9}{4} \times \frac{4}{3} = 3$$

Eg. 2.65 in text
$x(n) = p^n u[n]$
$r_{xx}(\ell) = \frac{1}{1-p^2}\,p^{|\ell|}$

$$r_{xx}[\ell] = 4\,\delta[\ell] - 3\,p^{\ell} u[\ell] - 3\,p^{-\ell} u[-\ell] + 3\,p^{|\ell|}$$

$$3\,p^{|\ell|} = 3\,p^{-\ell} u[-\ell] + 3\,p^{\ell} u[\ell] - 3\,\delta[\ell]$$

Thus, $r_{xx}[\ell] = \delta[\ell]$



(c) Determine and plot the autocorrelation sequence $r_{zz}[\ell]$ for $z[n]$ defined in terms of $x[n]$ in Eqn. (1) as below.

$$z[n] = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{\sqrt{2}}\right)}x[n]$$

Proved in class and on old exams (Fall 2006 Key Problem 1)

if $y[n] = e^{j(\omega_0 n+\theta)}x[n]$,

then $r_{yy}[\ell] = e^{j\omega_0 \ell}\, r_{xx}[\ell]$

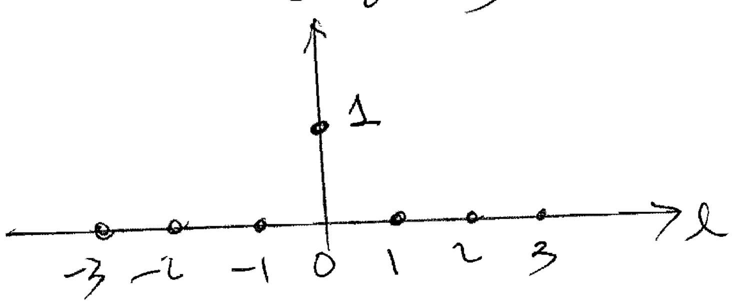
Applying here:

$$r_{zz}[\ell] = e^{j\frac{\pi}{2}\ell}\, r_{xx}[\ell]$$

$$= (j)^{\ell}\, r_{xx}[\ell]$$

$$= (j)^{\ell}\, \delta[\ell]$$

$$= \delta[\ell]$$



(d) Determine and plot the autocorrelation sequence $r_{zz}[\ell]$ for $z[n]$ defined in terms of $x[n]$ in Eqn. (1) below.

$$z[n] = x[2-n]$$

define: $y[n] = x[n+2]$

time-shift doesn't affect autocorrelation (proved on old exam Fall 2006 Key Problem 1)

$$r_{yy}[\ell] = r_{xx}[\ell]$$

next, define $z[n] = y[-n]$

proved in part (a), time-reversal doesn't affect autocorrelation, Thus.

$$r_{zz}[\ell] = r_{yy}[\ell]$$
$$= r_{xx}[\ell]$$

same plot as part (c)

(e) Consider a simple radar example where there are two echoes such that the received signal may be expressed in terms of $x[n]$ in Eqn. (1) as

$$y[n] = x[n-3] + \frac{1}{2}x[n-5] \tag{2}$$

Compute and plot the cross-correlation sequence $r_{yx}[\ell]$ given the input sequence $x[n]$ defined in Eqn. (1) above.

From class, we learned:

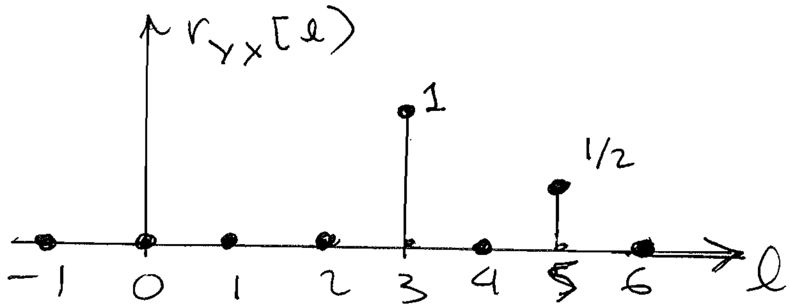
$$r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$$

where:

$$h[\ell] = \delta[n-3] + \frac{1}{2}\delta[n-5]$$

$$= r_{xx}[\ell-3] + \frac{1}{2} r_{xx}[\ell-5]$$

$$= \delta[\ell-3] + \frac{1}{2}\delta[\ell-5]$$



(f) Recall the simple radar example described by Eqn (2). Compute and plot the **auto-correlation** sequence for the **output** $r_{yy}[\ell]$ given the input sequence $x[n]$ defined in Eqn. (1). Can echo delays be determined from the autocorrelation of the output $r_{yy}[\ell]$?

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

$$r_{xx}[\ell] = \delta[\ell] \qquad r_{hh}[\ell] = h[\ell] * h[-\ell]$$

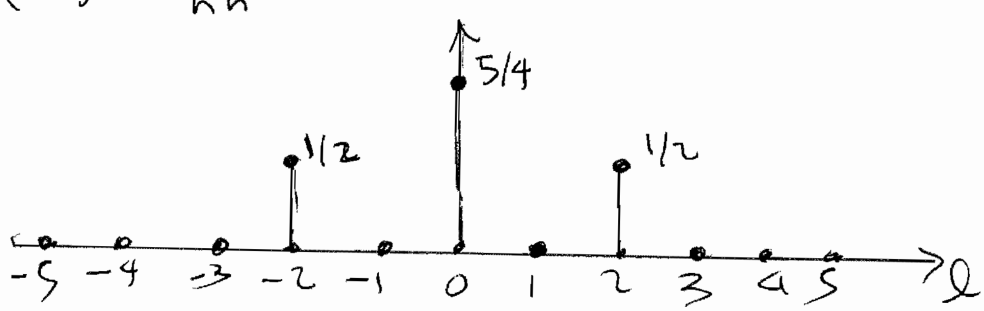
$$= \{\delta[\ell-3] + .5\,\delta[\ell-5]\} * \{\delta[-\ell-3] + .5\,\delta[-\ell-5]\}$$

$$= \{\delta[\ell-3] + .5\,\delta[\ell-5]\} * \{\delta[\underbrace{\ell+3}_{\ell-(-3)}] + .5\,\delta[\underbrace{\ell+5}_{\ell-(-5)}]\}$$

$$\boxed{\text{Since } \delta[n-n_1] * \delta[n-n_2] = \delta[n-(n_1+n_2)]}$$

$$r_{hh}[\ell] = \delta[\ell] + \tfrac{1}{4}\delta[\ell] + \tfrac{1}{2}\delta[\ell-2] + \tfrac{1}{2}\delta[\ell-2]$$

$$r_{xx}[\ell] = r_{hh}[\ell] \quad \text{since} \quad r_{xx}[\ell] = \delta[\ell]$$



Cannot determine delays from $r_{yy}[\ell]$

Problem 2. [40 points] Consider the causal DT LTI system described the difference equation below which is used for all four parts of this problem.

$$y[n] = j\, y[n-1] + x[n] - x[n-4]$$

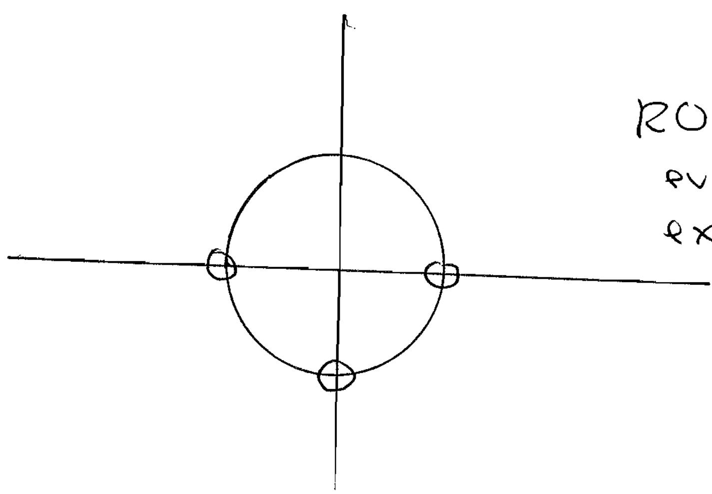
(a) Determine the transfer function $H(z)$ for this system and plot the pole-zero diagram. Show the region of convergence.

$$H(z) = \frac{1 - z^{-4}}{1 - j z^{-1}} = \frac{z}{z^4} \frac{z^4 - 1}{z - j} \qquad j = e^{j\frac{\pi}{2}}$$

zeroes at $z = 1, -1, j, -j$

pole at $z = j$

pole-zero cancellation at $z = j$



ROC is everywhere except $z = 0$

(b) Determine the impulse response of the system $h[n]$. You can either list the value of $h[n]$ for each n or write a closed-form expression for $h[n]$ that works for all n.

since pole-zero cancellation at $z = j$

$$H(z) = z^{-3}(z-1)(z+1)(z+j)$$

$$= z^{-3}(z^2-1)(z+j)$$

$$= z^{-3}(z^3 + jz^2 - z - j)$$

$$= 1 + jz^{-1} - z^{-2} - jz^{-3}$$

$$h[n] = \{\underset{\uparrow}{1}, j, -1, -j\}$$

$$= e^{j\frac{\pi}{2}n}\{u[n] - u[n-4]\}$$

from class

(c) Determine a closed-form expression for the frequency response of the system $H(\omega)$ equal to the DTFT of $h[n]$. Plot the magnitude $|H(\omega)|$ over $-\pi < \omega < \pi$. You can provide a rough sketch, but you need to clearly indicate the frequencies where $|H(\omega)| = 0$ and the frequency where $|H(\omega)|$ reaches its peak value.

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{1 - e^{-j4\omega}}{1 - j e^{-j\omega}}$$

$$= \frac{1 - e^{-j4\omega}}{1 - e^{-j(\omega - \frac{\pi}{2})}} \qquad j = e^{j\frac{\pi}{2}}$$

$$= \frac{e^{-j2\omega}\left(e^{j2\omega} - e^{-j2\omega}\right)}{e^{-j\frac{1}{2}(\omega - \frac{\pi}{2})}\left(e^{j\frac{1}{2}(\omega - \frac{\pi}{2})} - e^{-j\frac{1}{2}(\omega - \frac{\pi}{2})}\right)} \; \frac{\frac{1}{2j}}{\frac{1}{2j}}$$

$$= \frac{e^{-j2\omega}}{e^{-j\frac{1}{2}(\omega - \frac{\pi}{2})}} \; \frac{\sin(2\omega)}{\sin\left(\frac{1}{2}(\omega - \frac{\pi}{2})\right)} \Rightarrow$$

$\sin(2\omega) = 0$ (numerator)

when $2\omega = m\pi$

m: integer

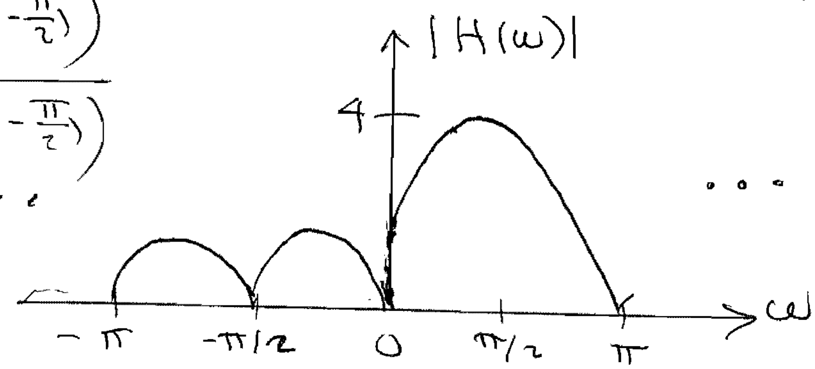
$\omega = m\frac{\pi}{2}$

$m \neq 1$ because of denominator

$$= \frac{e^{-j2(\omega - \frac{\pi}{2})} \sin\left(2(\omega - \frac{\pi}{2})\right)}{e^{-j\frac{1}{2}(\omega - \frac{\pi}{2})} \sin\left(\frac{1}{2}(\omega - \frac{\pi}{2})\right)}$$

$(-1)(-1) = 1$ as done in class

$$= e^{-j\frac{3}{2}(\omega - \frac{\pi}{2})} \frac{\sin\left(2(\omega - \frac{\pi}{2})\right)}{\sin\left(\frac{1}{2}(\omega - \frac{\pi}{2})\right)}$$



(d) Determine a closed-form expression for the output $y[n]$ when the input is the following sum of sinewaves turned-on forever:

$$x[n] = 1 + 2(-j)^n + 3(j)^n + 4(-1)^n$$

$$y[n] = \underbrace{H(0)}_{0} 1 + 2\underbrace{H\left(-\tfrac{\pi}{2}\right)}_{0}(-j)^n + 3\underbrace{H\left(\tfrac{\pi}{2}\right)}_{4}(j)^n + \underbrace{H(\pi)}_{0} 4(-1)^n$$

$$= 12(j)^n$$