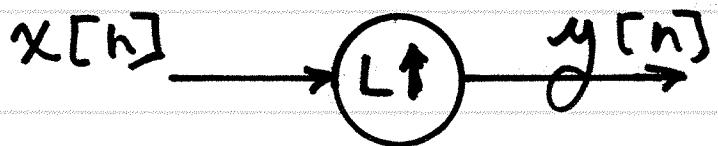


# • Key formulas for Multirate Analysis

①

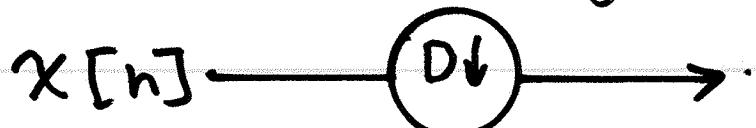


Insert  $L-1$  zeros  
between each successive  
values of  $x[n]$

Time Domain:  $y[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-kL]$

Frequency Domain:  $Y(\omega) = X(L\omega)$

Decimator:  $y[n] = x[Dn]$



Frequency Domain:  $Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - k2\pi}{D}\right)$

- Related to the polyphase filters, we ② encountered terms like:

$$h_L[n] = h[Ln + l] \xrightleftharpoons{DTFT} H_L(\omega) = ?$$

- First consider:  $g_L[n] = h[n+l]$

$$\Rightarrow G_L(\omega) = e^{j\omega l} H(\omega)$$

- Next:  $h_L[n] = g_L[Ln] = h[Ln + l]$

$$\text{Thus: } H_L(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} G_L\left(\frac{\omega - k2\pi}{L}\right)$$

Substitute:  $G_L(\omega) = e^{j\omega l} H(\omega)$

$$H_L(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} e^{j \frac{(w - k\pi)L}{L}} + \left( \frac{w - k\pi}{L} \right) \quad (3)$$

$$= \left\{ \frac{1}{L} \sum_{k=0}^{L-1} e^{-j \frac{k\pi L}{L}} H \left( \frac{w - k\pi}{L} \right) \right\} e^{j \frac{\pi L}{L} w}$$

If ideal case :  $h[n] = \frac{\sin(\frac{\pi}{L}n)}{\frac{\pi}{L}n}$

in which case :

$$H_L(\omega) = e^{j \frac{\pi L}{L} \omega} \text{ for } -\pi < \omega < \pi$$

corresponding to shift by fractional amount

$(l/L) T_s$  back in the time domain

# Alternative Derivation of DTFT of $h[nL + \ell]$

①

Note re: efficient polyphase implementation of up-sampling by a factor of  $L$ :

$$h[nL + \ell] \xleftarrow{\text{DTFT}} H_L(\omega) = ?$$

Recall:

$$\sum_{k=0}^{L-1} \frac{1}{L} e^{j \frac{2\pi}{L} k n} = \sum_{k=-\infty}^{\infty} \delta[n - kL]$$

Thus:

$$\frac{1}{L} \sum_{k=0}^{L-1} e^{j \frac{2\pi}{L} k (n-\ell)} = \begin{cases} 1, & \text{when } n = ml + \ell \\ 0, & \text{otherwise} \end{cases}$$

$$H_2(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\omega}$$

(2)

$$n' = L n + l \Rightarrow n = \frac{n' - l}{L}$$

$$H_2(\omega) = \sum_{n'=Ln+l} h[n'] e^{-jn'\omega/L}$$

$$= \sum_{n'=-\infty}^{\infty} \frac{1}{L} \sum_{k=0}^{L-1} e^{j \frac{2\pi k}{L} (n' - l)} h[n'] e^{-jn'\omega/L}$$

$n' = -\infty$

$k=0$

$$= \sum_{k=0}^{L-1} \frac{1}{L} e^{j \frac{2\pi k}{L} l} \sum_{n=-\infty}^{\infty} h[n] e^{-jn\frac{\omega}{L}(l - 2\pi k)}$$

$k=0$

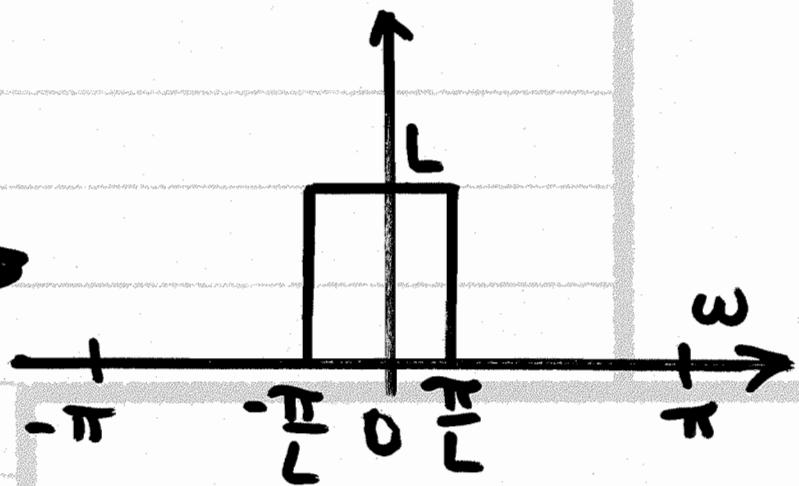
(3)

$$H_L(\omega) = \left\{ \sum_{k=0}^{L-1} \frac{1}{L} e^{-j \frac{2\pi L k}{L} \omega} H\left(\frac{\omega - k\pi}{L}\right) \right\} e^{j \omega \frac{\pi}{L}}$$

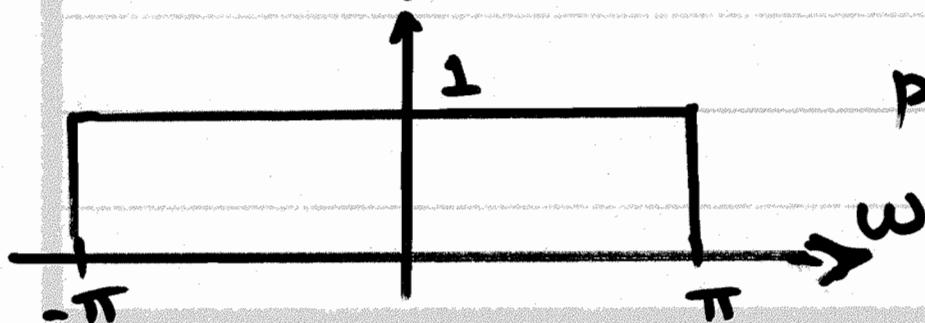
is the DTFT of  $h[n+L]$

Consider special case:

$$h[n] = L \frac{\sin\left(\frac{\pi}{L}n\right)}{\pi n} \quad \xleftrightarrow{\text{DTFT}}$$



Then:  $\frac{1}{L} H\left(\frac{\omega}{L}\right)$



period =  $L \cdot 2\pi$

The other values of  $k$  don't contribute  
 $\Rightarrow -\pi < \omega < \pi \Rightarrow$  just serve to make  
 $H_k(\omega)$  be periodic with period  $2\pi$

Thus, for this special case, for  $|\omega| < \pi$

$$H_k(\omega) = e^{j\omega \frac{k}{L}} \quad \text{for } |\omega| < \pi$$

recall:  $x[n - n_0] \xleftrightarrow{\text{DTFT}} X(\omega) e^{-j\omega n_0}$   
 $= x_a(nT_s - n_0 T_s)$

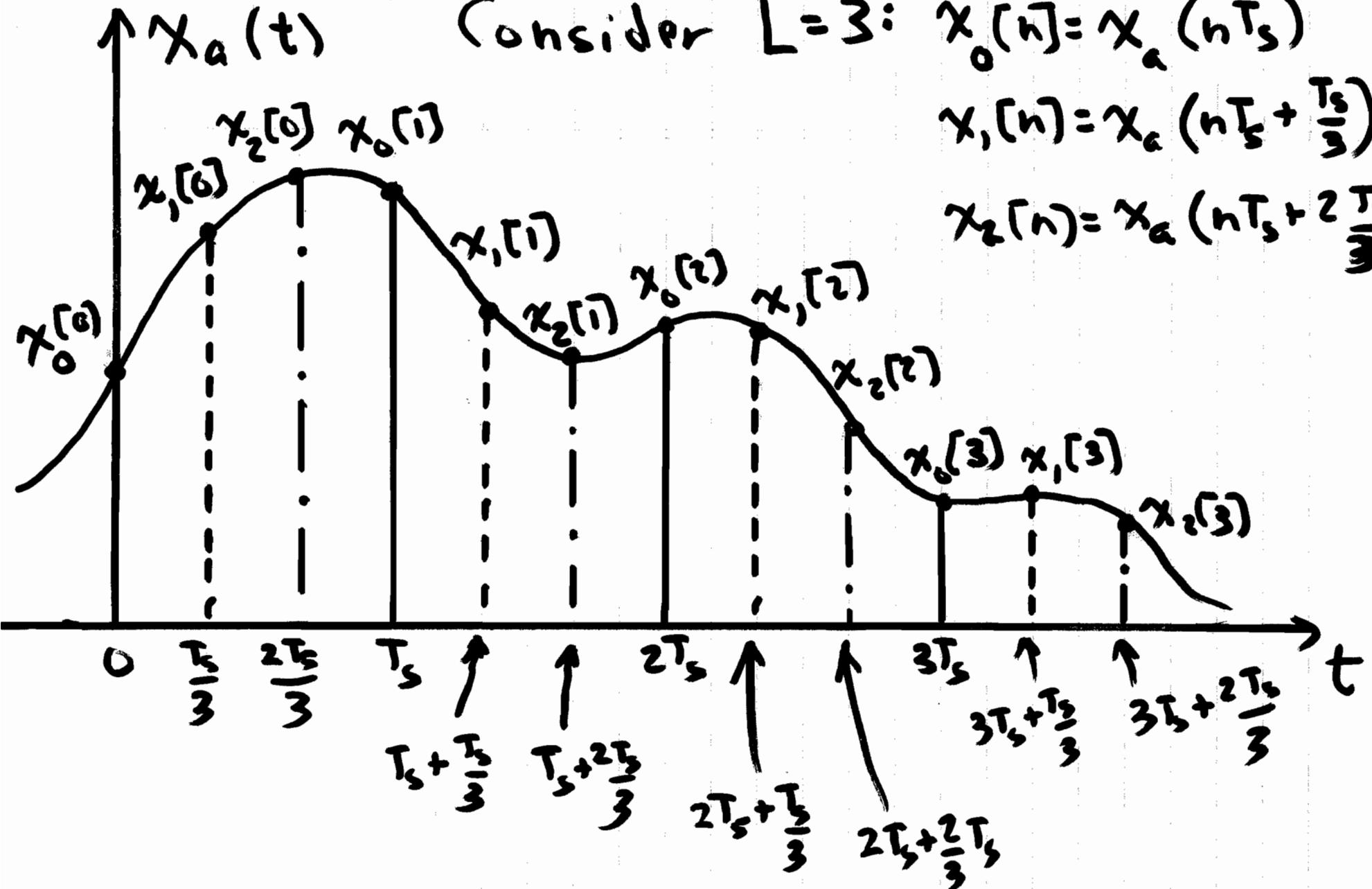
thus:  $e^{j\omega \frac{k}{L}}$  translates into a time-shift to  
 the left of  $\frac{k}{L}T_s$  in the analog domain  
 $\Rightarrow$  a fraction of a sample time-shift

(5)

Consider  $L=3$ :  $x_0[n] = x_a(nT_s)$ 

$$x_1[n] = x_a\left(nT_s + \frac{T_s}{3}\right)$$

$$x_2[n] = x_a\left(nT_s + 2\frac{T_s}{3}\right)$$



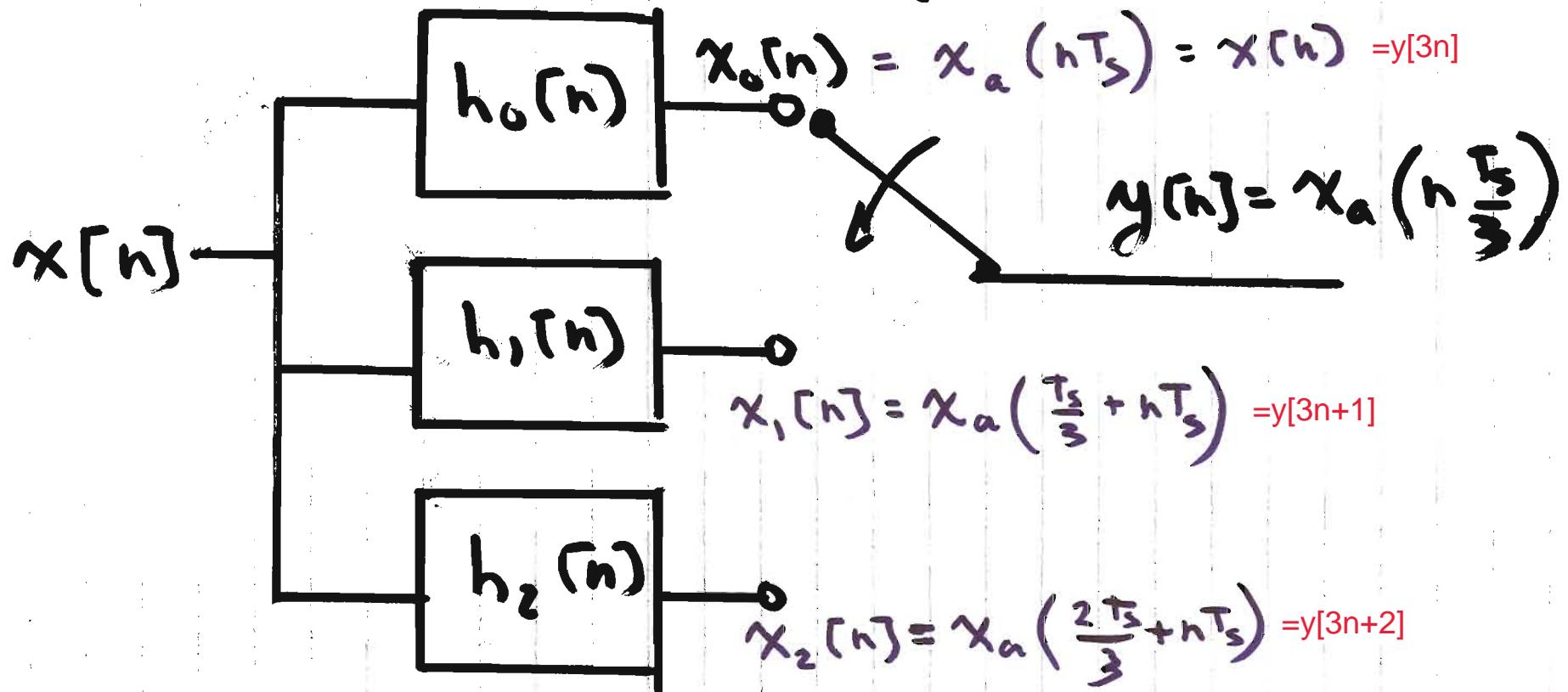
$$h[n] = 3 \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$$

$$h_0[n] = h[0n]$$

$$h_1[n] = h[1n]$$

$$h_2[n] = h[2n]$$

(6)



7

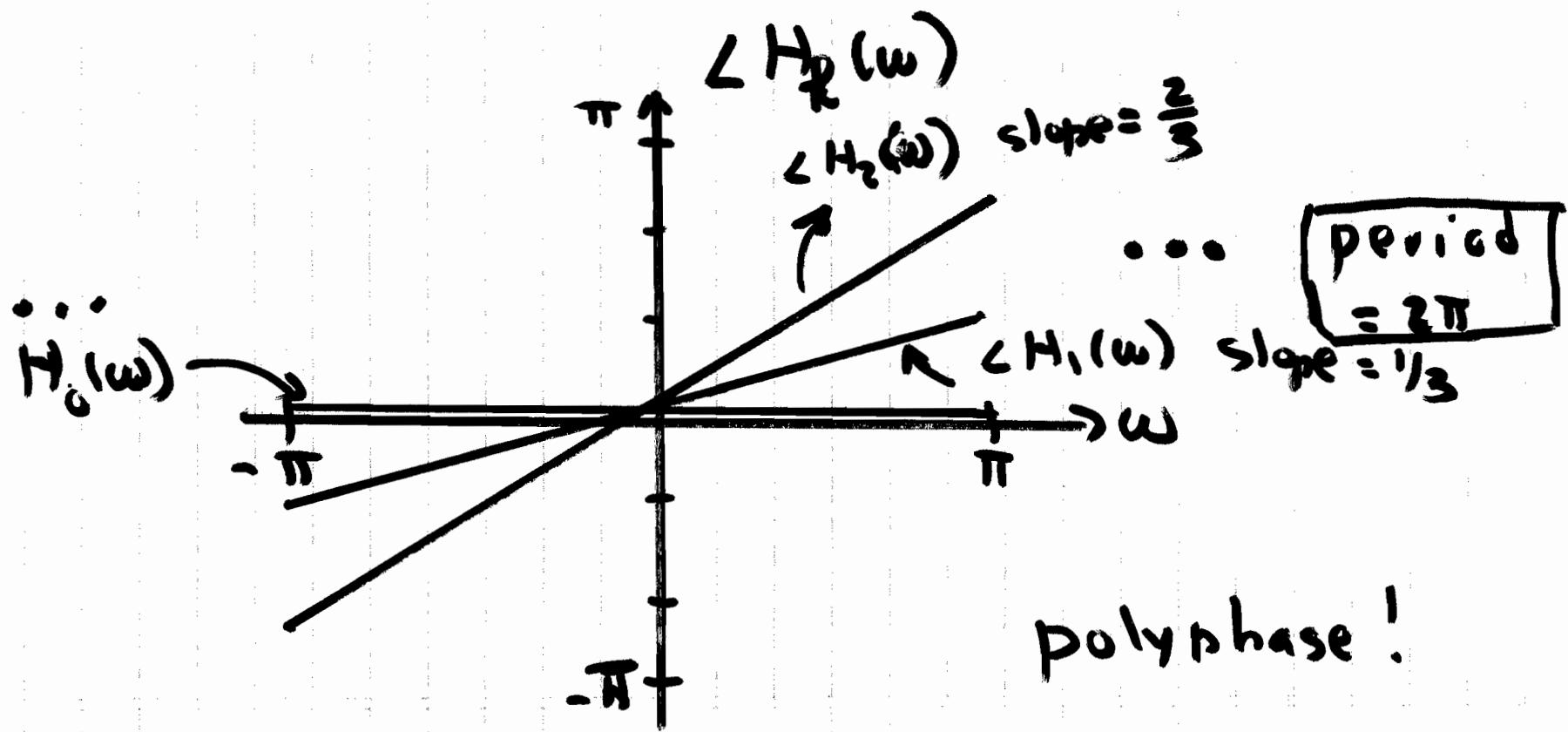
for  $|\omega| < \pi$  :  $H_0(\omega) = 1$

$$H_1(\omega) = e^{j\frac{1}{3}\omega}$$

$$H_2(\omega) = e^{j\frac{2}{3}\omega}$$

$$\Rightarrow \angle H_1(\omega) = \frac{\omega}{3}$$

$$\Rightarrow \angle H_2(\omega) = \frac{2}{3}\omega$$



## Note on Fractional Time-Shift

- Recall ideal Reconstruction Formula

$$x_a(t) = \sum_{k=-\infty}^{\infty} x_a(kT_s) \frac{\sin\left(\frac{\pi}{T_s}(t-kT_s)\right)}{\frac{\pi}{T_s}(t-kT_s)}$$

- Evaluate at :  $t = \frac{\ell}{L} T_s + nT_s$   $\ell \in \{0, 1, \dots, L-1\}$   
 $L = \text{integer}$

$$\begin{aligned}
 y[n] &= x_a\left(\frac{\ell}{L} T_s + nT_s\right) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\frac{\pi}{T_s}\left(\frac{\ell}{L} T_s + nT_s - kT_s\right)\right)}{\frac{\pi}{T_s}\left(\frac{\ell}{L} T_s + nT_s - kT_s\right)} \\
 &= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\pi\left(\frac{\ell}{L} + n - k\right)\right)}{\pi\left(\frac{\ell}{L} + n - k\right)} h[n] \\
 &= x[n] * h[n] \\
 &= x_a\left(\frac{\ell}{L} T_s + nT_s\right)
 \end{aligned}$$