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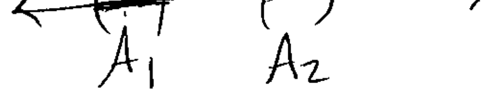
The Probability Measure

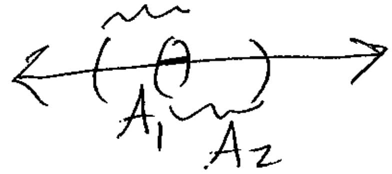
We use a set function $P: \mathcal{F} \to \mathbb{R}$ to assign a prob. $P(A)$ to every $A \in \mathcal{F}$. A valid prob. measure P is one that satisfies the following "axioms of probability":

1. $P(A) \geq 0 \quad \forall A \in \mathcal{F}$
2. $P(S) = 1$
3. If A_1 and A_2 are disjoint events (i.e., $A_1 \cap A_2 = \emptyset$), then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

for example, measure a temp.



$A_1 \cap A_2 = \emptyset$



$A_1 \cap A_2 \neq \emptyset$

$P(A_1 \cup A_2)$ not necess. equal to $P(A_1) + P(A_2)$

Note that by induction, we must have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

for finite $n \geq 2$, if $A_1, \ldots, A_n$ are disjoint

(4). If $A_1, A_2, \ldots$ are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Note that Axiom 4 $\Rightarrow$ Axiom 3, but if there are only a finite number of events, you only need to show Axioms 1–3.

Some Important Properties that follow from the Axioms:

(1) $P(\phi) = 0$ for any $(S, \mathcal{F}, P)$

Proof. $P(\mathcal{S}) = 1$ (Ax. 2)

Since $\mathcal{S} \cap \phi = \phi$, we have

$$P(\mathcal{S} \cup \phi) = P(\mathcal{S}) + P(\phi) \quad \text{(Ax. 3)}$$

But $\mathcal{S} \cup \phi = \mathcal{S}$, so

$$P(\mathcal{S}) = P(\mathcal{S}) + P(\phi)$$

$$1 = 1 + P(\phi) \Rightarrow$$

$$P(\phi) = 0.$$

Important note:

If $A = \phi$, then $P(A) = 0$,

but

if $P(A) = 0$, A might not be ϕ

(2) $P(A^c) = 1 - P(A) \quad \forall A \in \mathcal{F}$

Proof. Can write ~~of~~ the ~~statemen~~ statement to be proved as

$$P(A^c) + P(A) = 1$$

Since A^c and A are disjoin

$P(A^c)+P(A) = P(A^c \cup A)$ (Ax. 2) 4

so $P(A^c)+P(A) = P(S) = 1$

3 $P(A) \leq P(B)$ if $A \subset B$.

Proof omitted

4 $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\forall A, B \in \mathcal{F}$

Proof omitted

Note that the counting approach and relative frequency approach to prob. lead to measure P that satisfy the axioms

- The counting approach

If S is finite, then $\forall A \in \mathcal{F}$, let

$$P_c(A) = \frac{|A|}{|S|}$$

where $|A|$ is the number

of elements in A.

- The relative frequency appr.
  Run an experiment n times for finite n. Then $\forall A \in \mathcal{F}$, let

  $$P_n(A) = \frac{n_A}{n}$$

  if A occurs n_A times.
  Can let $P(A) = \lim_{n \to \infty} P_n(A)$

It can be shown that $P_c(A)$ and $P_n(A)$, $\forall n$, satisfy the axioms. The form of $P(A)$ for the rel. freq. approach is the subject of the LLNs.

Statistical Independence of Events

Defn. For any $(S, \mathcal{F}, P)$ and events $A, B \in \mathcal{F}$, A and B are <u>statistically independent</u> if

$$P(A \cap B) = P(A)P(B)$$

Defn. Events $A_1, \ldots, A_n$ are (stat.) independent if for any $k = 2, \ldots, n$ and any $1 \leq j_1, \ldots, j_k \leq n$, for any finite n,

$$P(A_{j_1} \cap \cdots \cap A_{j_k}) = \prod_{i=1}^{k} P(A_{j_i})$$

Note that for any n, there are $2^n - (n+1)$ combinations of events for which this equality must be shown