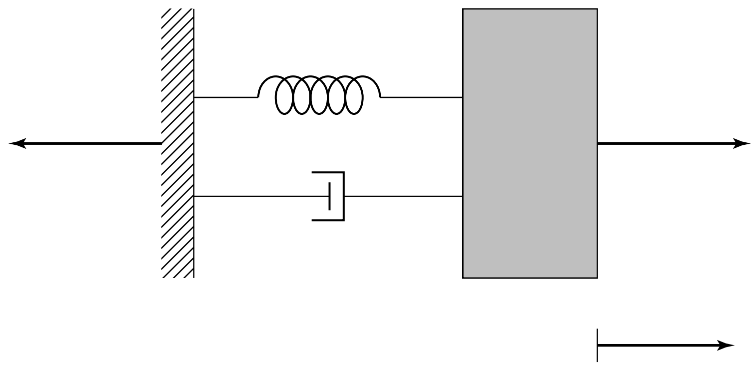
CHAPTER 4

Tuned Mass Damper Systems



4.1 INTRODUCTION

A tuned mass damper (TMD) is a device consisting of a mass, a spring, and a damper that is attached to a structure in order to reduce the dynamic response of the structure. The frequency of the damper is tuned to a particular structural frequency so that when that frequency is excited, the damper will resonate out of phase with the structural motion. Energy is dissipated by the damper inertia force acting on the structure. The TMD concept was first applied by Frahm in 1909 (Frahm, 1909) to reduce the rolling motion of ships as well as ship hull vibrations. A theory for the TMD was presented later in the paper by Ormondroyd and Den Hartog (1928), followed by a detailed discussion of optimal tuning and damping parameters in Den Hartog's book on mechanical vibrations (1940). The initial theory was applicable for an undamped SDOF system subjected to a sinusoidal force excitation. Extension of the theory to damped SDOF systems has been investigated by numerous researchers. Significant contributions were made by Randall et al. (1981), Warburton (1981, 1982), Warburton and Ayorinde (1980), and Tsai and Lin (1993).

This chapter starts with an introductory example of a TMD design and a brief description of some of the implementations of tuned mass dampers in building structures. A rigorous theory of tuned mass dampers for SDOF systems subjected to harmonic force excitation and harmonic ground motion is discussed next. Various cases, including an undamped TMD attached to an undamped SDOF system, a damped TMD attached to an undamped SDOF system, and a damped TMD attached to a damped SDOF system, are considered. Time history responses for a range of SDOF systems connected to optimally tuned TMD and subjected to harmonic and seismic excitations are presented. The theory is then extended to MDOF systems, where the TMD is used to dampen out the vibrations of a specific mode. An assessment of the optimal placement locations of TMDs in building structures is included. Numerous examples are provided to illustrate the level of control that can be achieved with such passive devices for both harmonic and seismic excitations.

4.2 AN INTRODUCTORY EXAMPLE

In this section, the concept of the tuned mass damper is illustrated using the two-mass system shown in Figure 4.1. Here, the subscript d refers to the *tuned mass damper*; the structure is idealized as a single degree of freedom system. Introducing the following notation

$$\omega^2 = \frac{k}{m} \tag{4.1}$$

$$c = 2\xi\omega m \tag{4.2}$$

$$\omega_d^2 = \frac{k_d}{m_d} \tag{4.3}$$

$$c_d = 2\xi_d\omega_d m_d \tag{4.4}$$

and defining $\bar{m}$ as the mass ratio,

$$\bar{m} = \frac{m_d}{m} \tag{4.5}$$

the governing equations of motion are given by

$$\text{Primary mass } (1+\bar{m})\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \frac{p}{m} - \bar{m}\ddot{u}_d \tag{4.6}$$

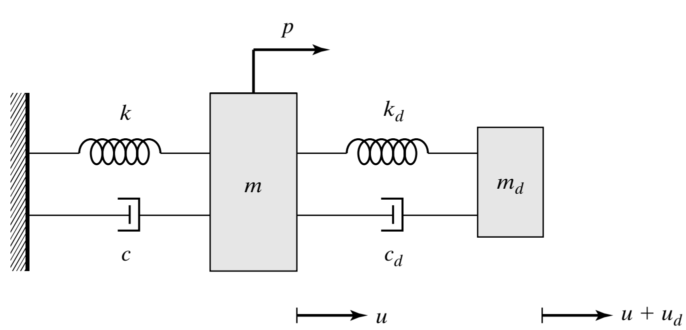


FIGURE 4.1: SDOF-TMD system.

$$\text{Tuned mass } \ddot{u}_d + 2\xi_d\omega_d\dot{u}_d + \omega_d^2 u_d = -\ddot{u} \tag{4.7}$$

The purpose of adding the mass damper is to limit the motion of the structure when it is subjected to a particular excitation. The design of the mass damper involves specifying the mass m_d, stiffness k_d, and damping coefficient c_d. The optimal choice of these quantities is discussed in Section 4.4. In this example, the *near-optimal* approximation for the frequency of the damper,

$$\omega_d = \omega \tag{4.8}$$

is used to illustrate the design procedure. The stiffnesses for this frequency combination are related by

$$k_d = \bar{m}k \tag{4.9}$$

Equation (4.8) corresponds to tuning the damper to the fundamental period of the structure.

Considering a periodic excitation,

$$p = \hat{p}\sin\Omega t \tag{4.10}$$

the response is given by

$$u = \hat{u}\sin(\Omega t + \delta_1) \tag{4.11}$$

$$u_d = \hat{u}_d\sin(\Omega t + \delta_1 + \delta_2) \tag{4.12}$$

where $\hat{u}$ and δ denote the displacement amplitude and phase shift, respectively. The critical loading scenario is the resonant condition, $\Omega = \omega$. The solution for this case has the following form:

$$\hat{u} = \frac{\hat{p}}{k\bar{m}}\sqrt{\frac{1}{1 + \left(\frac{2\xi}{\bar{m}} + \frac{1}{2\xi_d}\right)^2}} \tag{4.13}$$

$$\hat{u}_d = \frac{1}{2\xi_d}\hat{u} \tag{4.14}$$

$$\tan\delta_1 = -\left[\frac{2\xi}{\bar{m}} + \frac{1}{2\xi_d}\right] \tag{4.15}$$

$$\tan\delta_2 = -\frac{\pi}{2} \tag{4.16}$$

Note that the response of the tuned mass is 90º out of phase with the response of the primary mass. This difference in phase produces the *energy dissipation* contributed by the damper inertia force.

The response for *no* damper is given by

$$\hat{u} = \frac{\hat{p}}{k}\left(\frac{1}{2\xi}\right) \tag{4.17}$$

$$\delta_1 = \left(-\frac{\pi}{2}\right) \tag{4.18}$$

To compare these two cases, we express Eq. (4.13) in terms of an equivalent damping ratio:

$$\hat{u} = \frac{\hat{p}}{k}\left(\frac{1}{2\xi_e}\right) \tag{4.19}$$

where

$$\xi_e = \frac{\overline{m}}{2}\sqrt{1 + \left(\frac{2\xi}{\overline{\overline{m}}} + \frac{1}{2\xi_d}\right)^2} \tag{4.20}$$

Equation (4.20) shows the relative contribution of the damper parameters to the total damping. Increasing the mass ratio magnifies the damping. However, since the added mass also increases, there is a practical limit on $\overline{m}$. Decreasing the damping coefficient for the damper also increases the damping. Noting Eq. (4.14), the relative displacement also increases in this case, and just as for the mass, there is a practical limit on the relative motion of the damper. Selecting the final design requires a compromise between these two constraints.

Example 4.1: Preliminary design of a TMD for a SDOF system

Suppose $\xi = 0$ and we want to add a tuned mass damper such that the equivalent damping ratio is 10%. Using Eq. (4.20), and setting $\xi_e = 0.1$, the following relation between $\overline{m}$ and ξ_d is obtained:

$$\frac{\overline{m}}{2}\sqrt{1 + \left(\frac{2\xi}{\overline{\overline{m}}} + \frac{1}{2\xi_d}\right)^2} = 0.1 \tag{1}$$

The relative displacement constraint is given by Eq. (4.14):

$$\hat{u}_d = \left(\frac{1}{2\xi_d}\right)\hat{u} \tag{2}$$

Combining Eq. (1) and Eq. (2) and setting $\xi = 0$ leads to

$$\frac{\overline{m}}{2}\sqrt{1+\left(\frac{\hat{u}_d}{\hat{u}}\right)^2} = 0.1 \tag{3}$$

Usually, $\hat{u}_d$ is taken to be an order of magnitude greater than $\hat{u}$. In this case, Eq. (3) can be approximated as

$$\frac{\overline{m}}{2}\left(\frac{\hat{u}_d}{\hat{u}}\right) \approx 0.1 \tag{4}$$

The generalized form of Eq. (4) follows from Eq. (4.20):

$$\overline{m} \approx 2\xi_e\left(\frac{1}{\hat{u}_d/\hat{u}}\right) \tag{5}$$

Finally, taking $\hat{u}_d = 10\hat{u}$ yields an estimate for $\overline{m}$:

$$\overline{m} = \frac{2(0.1)}{10} = 0.02 \tag{6}$$

This magnitude is typical for $\overline{m}$. The other parameters are

$$\xi_d = \frac{1}{2}\left(\frac{\hat{u}}{\hat{u}_d}\right) = 0.05 \tag{7}$$

and from Eq. (4.9)

$$k_d = \overline{m}k = 0.02k \tag{8}$$

It is important to note that with the addition of only 2% of the primary mass, we obtain an effective damping ratio of 10%. The negative aspect is the large relative motion of the damper mass; in this case, 10 times the displacement of the primary mass. How to accommodate this motion in an actual structure is an important design consideration.

A description of some applications of tuned mass dampers to building structures is presented in the following section to provide additional background on this type of device prior to entering into a detailed discussion of the underlying theory.

4.3 EXAMPLES OF EXISTING TUNED MASS DAMPER SYSTEMS

Although the majority of applications have been for mechanical systems, tuned mass dampers have been used to improve the response of building structures under wind excitation. A short description of the various types of dampers and several building structures that contain tuned mass dampers follows.

4.3.1 Translational Tuned Mass Dampers

Figure 4.2 illustrates the typical configuration of a unidirectional translational tuned mass damper. The mass rests on bearings that function as rollers and allow the mass to translate laterally relative to the floor. Springs and dampers are inserted between the mass and the adjacent vertical support members, which transmit the lateral "out-of-phase" force to the floor level and then into the structural frame. Bidirectional translational dampers are configured with springs/dampers in two orthogonal directions and provide the capability for controlling structural motion in two orthogonal planes. Some examples of early versions of this type of damper are described next.

- ***John Hancock Tower*** (*Engineering News Record*, Oct. 1975)

Two dampers were added to the 60-story John Hancock Tower in Boston to reduce the response to wind gust loading. The dampers are placed at opposite ends of the fifty-eighth story, 67 m apart, and move to counteract sway as well as twisting due to the shape of the building. Each damper weighs 2700 kN and consists of a lead-filled steel box about 5.2 m square and 1 m deep that rides on a 9-m-long steel plate. The lead-filled weight, laterally restrained by stiff springs anchored to the interior columns of the building and controlled by servo-hydraulic cylinders, slides back and forth on a hydrostatic bearing consisting of a thin layer of oil forced through holes in the steel plate. Whenever the horizontal acceleration exceeds 0.003*g* for two consecutive cycles, the system is automatically activated. This system was designed and manufactured by LeMessurier Associates/SCI in association with MTS System Corp., at a cost of around 3 million dollars, and is expected to reduce the sway of the building by 40 to 50%.

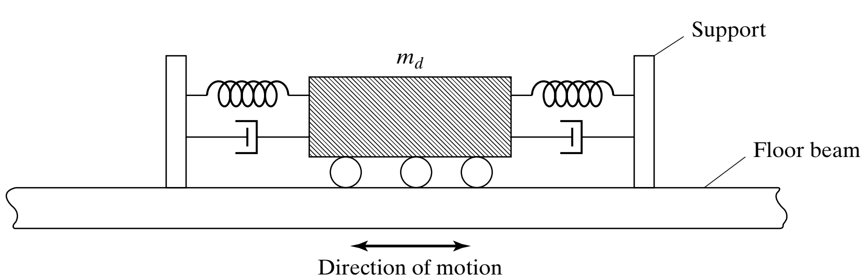


FIGURE 4.2: Schematic diagram of a translational tuned mass damper.

- ***Citicorp Center*** (*Engineering News Record,* Aug. 1975, McNamara 1977, Petersen 1980)

The Citicorp (Manhattan) TMD was also designed and manufactured by LeMessurier Associates/SCI in association with MTS System Corp. This building is 279 m high and has a fundamental period of around 6.5 s with an inherent damping ratio of 1% along each axis. The Citicorp TMD, located on the sixty-third floor in the crown of the structure, has a mass of 366 Mg, about 2% of the effective modal mass of the first mode, and was 250 times larger than any existing tuned mass damper at the time of installation. Designed to be biaxially resonant on the building structure with a variable operating period of 6.25 s ±20%, adjustable linear damping from 8 to 14%, and a peak relative displacement of ±1.4 m, the damper is expected to reduce the building sway amplitude by about 50%. This reduction corresponds to increasing the basic structural damping by 4%. The concrete mass block is about 2.6 m high with a plan cross section of 9.1 m by 9.1 m and is supported on a series of twelve 60-cm-diameter hydraulic pressure-balanced bearings. During operation, the bearings are supplied oil from a separate hydraulic pump, which is capable of raising the mass block about 2 cm to its operating position in about 3 minutes. The damper system is activated automatically whenever the horizontal acceleration exceeds 0.003*g* for two consecutive cycles and will automatically shut itself down when the building acceleration does not exceed 0.00075*g* in either axis over a 30-minute interval. LeMessurier estimates Citicorp's TMD, which cost about 1.5 million dollars, saved 3.5 to 4 million dollars. This sum represents the cost of some 2800 tons of structural steel that would have been required to satisfy the deflection constraints.

- ***Canadian National Tower*** (*Engineering News Record*, 1976)

The 102-m steel antenna mast on top of the Canadian National Tower in Toronto (553 m high including the antenna) required two lead dampers to prevent the antenna from deflecting excessively when subjected to wind excitation. The damper system consists of two doughnut-shaped steel rings, 35 cm wide, 30 cm deep, and 2.4 m and 3 m in diameter, located at elevations 488 m and 503 m. Each ring holds about 9 metric tons of lead and is supported by three steel beams attached to the sides of the antenna mast. Four bearing universal joints that pivot in all directions connect the rings to the beams. In addition, four separate hydraulically activated fluid dampers mounted on the side of the mast and attached to the center of each universal joint dissipate energy. As the lead-weighted rings move back and forth, the hydraulic damper system dissipates the input energy and reduces the tower's response. The damper system was designed by Nicolet, Carrier, Dressel, and Associates, Ltd., in collaboration with Vibron Acoustics, Ltd. The dampers are tuned to the second and fourth modes of vibration in order to minimize antenna bending loads; the first and third modes have the same characteristics as the prestressed concrete structure supporting the antenna and did not require additional damping.

- ***Chiba Port Tower*** (Kitamura et al., 1988)

Chiba Port Tower (completed in 1986) was the first tower in Japan to be equipped with a TMD. Chiba Port Tower is a steel structure 125 m high weighing 1950 metric tons and having a rhombus-shaped plan with a side length of 15 m. The first and second mode periods are 2.25 s and 0.51 s, respectively for the *x* direction and 2.7 s and 0.57 s for the *y* direction. Damping for the fundamental mode is estimated at 0.5%. Damping ratios proportional to frequencies were assumed for the higher modes in the analysis. The purpose of the TMD is to increase damping of the first mode for both the *x* and *y* directions. Figure 4.3 shows the damper system. Manufactured by Mitsubishi Steel Manufacturing Co., Ltd., the damper has mass ratios with respect to the modal mass of the first mode of about 1/120 in the *x* direction and 1/80 in the *y* direction; periods in the *x* and *y* directions of 2.24 s and 2.72 s, respectively; and a damper damping ratio of 15%. The maximum relative displacement of the damper with respect to the tower is about ± 1 m in each direction. Reductions of around 30 to 40% in the displacement of the top floor and 30% in the peak bending moments are expected.

The early versions of TMDs employ complex mechanisms for the bearing and damping elements, have relatively large masses, occupy considerable space, and are quite expensive. Recent versions, such as the scheme shown in Figure 4.4, have been designed to minimize these limitations. This scheme employs a multiassemblage of elastomeric rubber bearings, which function as shear springs, and bitumen rubber compound (BRC) elements, which provide viscoelastic damping capability. The device is compact in size, requires unsophisticated controls, is multidirectional, and is easily assembled and modified. Figure 4.5 shows a full-scale damper being subjected to dynamic excitation by a shaking table. An actual installation is contained in Figure 4.6.

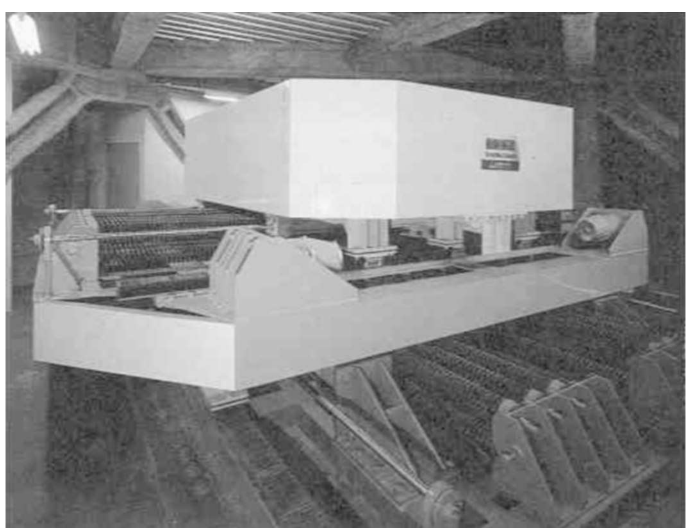


FIGURE 4.3: Tuned mass damper for Chiba-Port Tower. (Courtesy of J. Connor.)

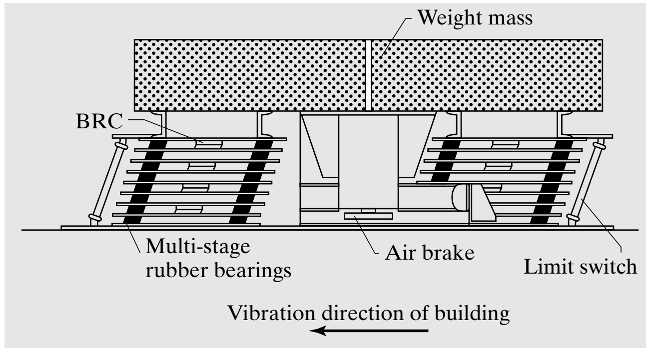


FIGURE 4.4: Tuned mass damper with spring and damper assemblage.

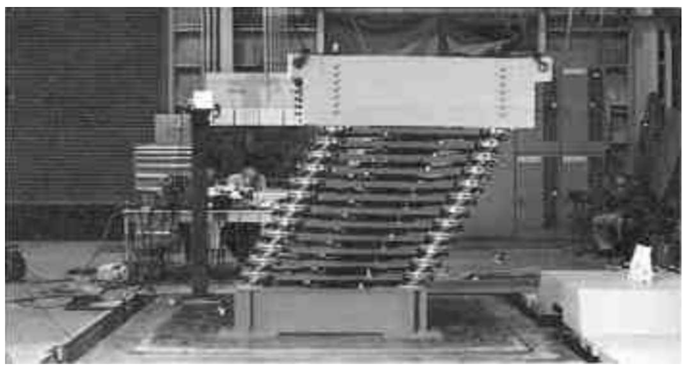


FIGURE 4.5: Deformed position—tuned mass damper. (Courtesy of J. Connor.)

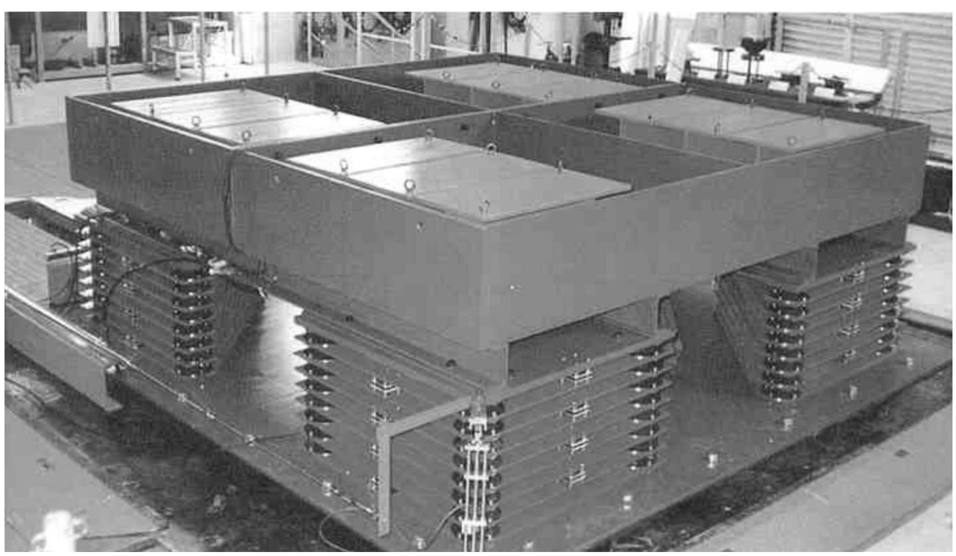


FIGURE 4.6: Tuned mass damper—Huis Ten Bosch Tower, Nagasaki. (Courtesy of J. Connor.)

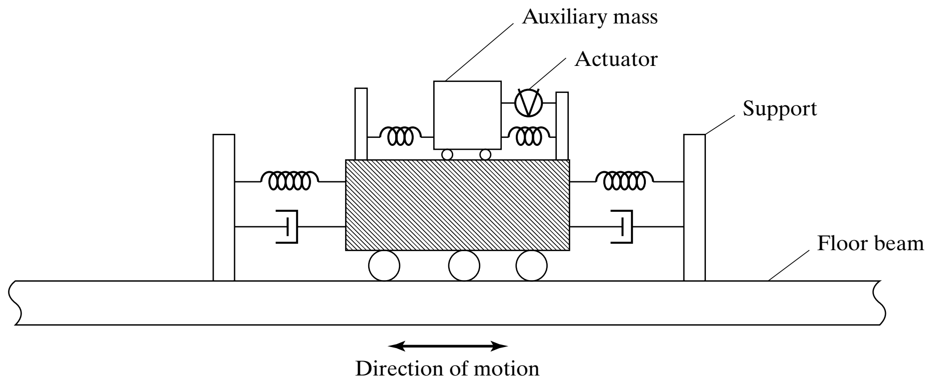


FIGURE 4.7: An active tuned mass damper configuration.

The effectiveness of a tuned mass damper can be increased by attaching an auxiliary mass and an actuator to the tuned mass and driving the auxiliary mass with the actuator such that its response is out of phase with the response of the tuned mass. Figure 4.7 illustrates this scheme. The effect of driving the auxiliary mass is to produce an additional force that complements the force generated by the tuned mass and therefore increases the equivalent damping of the TMD (we can obtain the same behavior by attaching the actuator directly to the tuned mass, thereby eliminating the need for an auxiliary mass). Since the actuator requires an external energy source, this system is referred to as an active tuned mass damper. The scope of this chapter is restricted to passive TMDs. Active TMDs are discussed in Chapter 6.

4.3.2 Pendulum Tuned Mass Damper

The problems associated with the bearings can be eliminated by supporting the mass with cables which allow the system to behave as a pendulum. Figure 4.8(a) shows a simple pendulum attached to a floor. Movement of the floor excites the pendulum. The relative motion of the pendulum produces a horizontal force that opposes the floor motion. This action can be represented by an equivalent SDOF system that is attached to the floor, as indicated in Figure 4.8(b).

The equation of motion for the horizontal direction is

$$T\sin\theta + \frac{W_d}{g}(\ddot{u} + \ddot{u}_d) = 0 \tag{4.21}$$

where T is the tension in the cable. When θ is small, the following approximations apply:

$$\begin{aligned} u_d &= L\sin\theta \approx L\theta \\ T &\approx W_d \end{aligned} \tag{4.22}$$

Introducing these approximations transforms Eq. (4.21) to

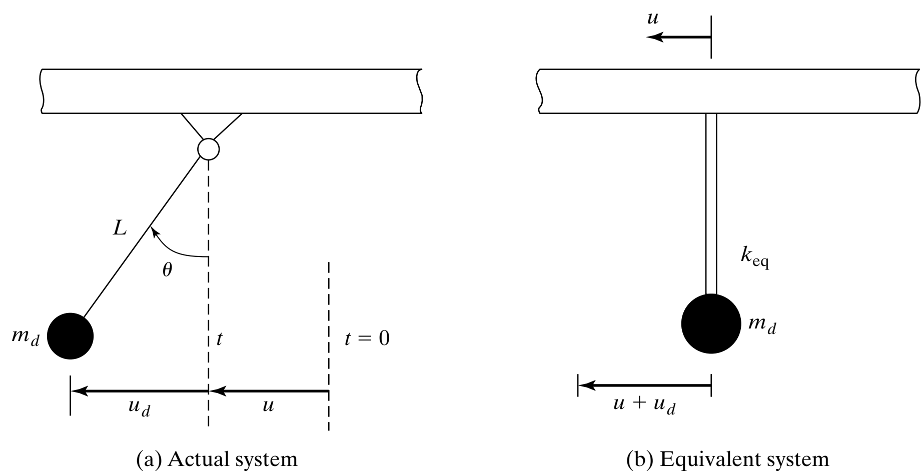


FIGURE 4.8: A simple pendulum tuned mass damper.

$$m_d \ddot{u}_d + \frac{W_d}{L} u_d = -m_d \ddot{u} \tag{4.23}$$

and it follows that the equivalent shear spring stiffness is

$$k_{eq} = \frac{W_d}{L} \tag{4.24}$$

The natural frequency of the pendulum is related to k_{eq} by

$$\omega_d^2 = \frac{k_{eq}}{m_d} = \frac{g}{L} \tag{4.25}$$

Noting Eq. (4.25), the natural period of the pendulum is

$$T_d = 2\pi\sqrt{\frac{L}{g}} \tag{4.26}$$

The simple pendulum tuned mass damper concept has a serious limitation. Since the period depends on L, the required length for large T_d may be greater than the typical story height. For instance, the length for $T_d = 5$ s is 6.2 meters whereas the story height is between 4 and 5 meters. This problem can be eliminated by resorting to the scheme illustrated in Figure 4.9. The interior rigid link magnifies the support motion for the pendulum and results in the following equilibrium equation:

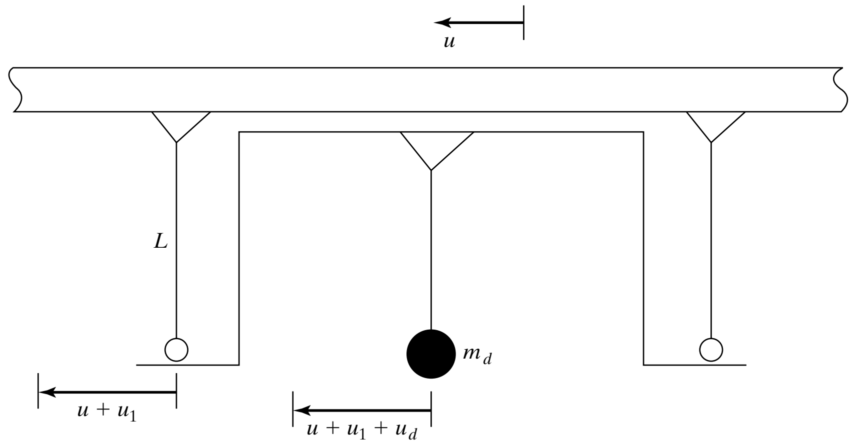


FIGURE 4.9: Compound pendulum.

$$m_d(\ddot{u} + \ddot{u}_1 + \ddot{u}_d) + \frac{W_d}{L}u_d = 0 \tag{4.27}$$

The rigid link moves in phase with the damper and has the same displacement amplitude. Then, taking $u_1 = u_d$ in Eq. (4.27) results in

$$m_d\ddot{u}_d + \frac{W_d}{2L}u_d = -\frac{m_d}{2}\ddot{u} \tag{4.28}$$

The equivalent stiffness is $W_d/2L$, and it follows that the effective length is equal to $2L$. Each additional link increases the effective length by L. An example of a pendulum-type damper is described next.

- ***Crystal Tower*** (Nagase and Hisatoku, 1990)

The tower, located in Osaka, Japan, is 157 m high and 28 m by 67 m in plan, weighs 44000 metric tons, and has a fundamental period of approximately 4 s in the north-south direction and 3 s in the east-west direction. A tuned pendulum mass damper was included in the early phase of the design to decrease the wind-induced motion of the building by about 50%. Six of the nine air cooling and heating ice thermal storage tanks (each weighing 90 tons) are hung from the top roof girders and used as a pendulum mass. Four tanks have a pendulum length of 4 m and slide in the north-south direction; the other two tanks have a pendulum length of about 3 m and slide in the east-west direction. Oil dampers connected to the pendulums dissipate the pendulum energy. Figure 4.10 shows the layout of the ice storage tanks that were used as damper masses. Views of the actual building and one of the tanks are presented in Figure 4.11 on page 230. The cost of this tuned mass damper system was around $350,000, less than 0.2% of the construction cost.

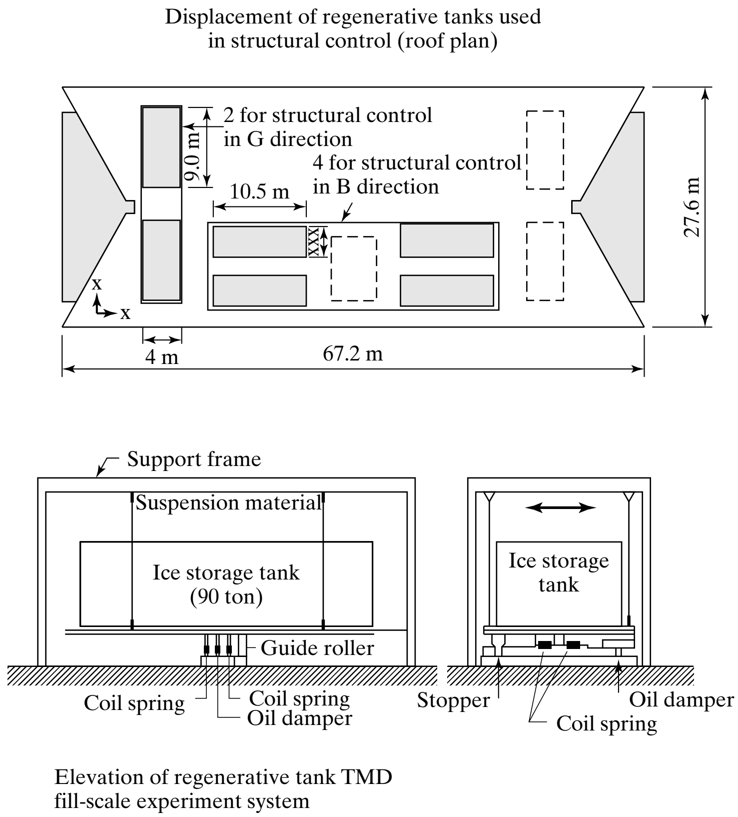


FIGURE 4.10: Pendulum damper layout—Crystal Tower. Takemaka Corporation.

A modified version of the pendulum damper is shown in Figure 4.12 on page 231. The restoring force provided by the cables is generated by introducing curvature in the support surface and allowing the mass to roll on this surface. The vertical motion of the weight requires an energy input. Assuming θ is small, the equations for the case where the surface is circular are the same as for the conventional pendulum with the cable length L, replaced with the surface radius R.

4.4 TUNED MASS DAMPER THEORY FOR SDOF SYSTEMS

In what follows, various cases ranging from fully undamped to fully damped conditions are analyzed and design procedures are presented.

4.4.1 Undamped Structure: Undamped TMD

Figure 4.13 shows a SDOF system having mass m and stiffness k, subjected to both external forcing and ground motion. A tuned mass damper with mass m_d and stiffness k_d is attached to the primary mass. The various displacement measures are u_g, the absolute ground motion; u, the relative motion between the primary mass and the ground; and u_d, the relative displacement between the damper and the primary mass. With this notation, the governing equations take the form

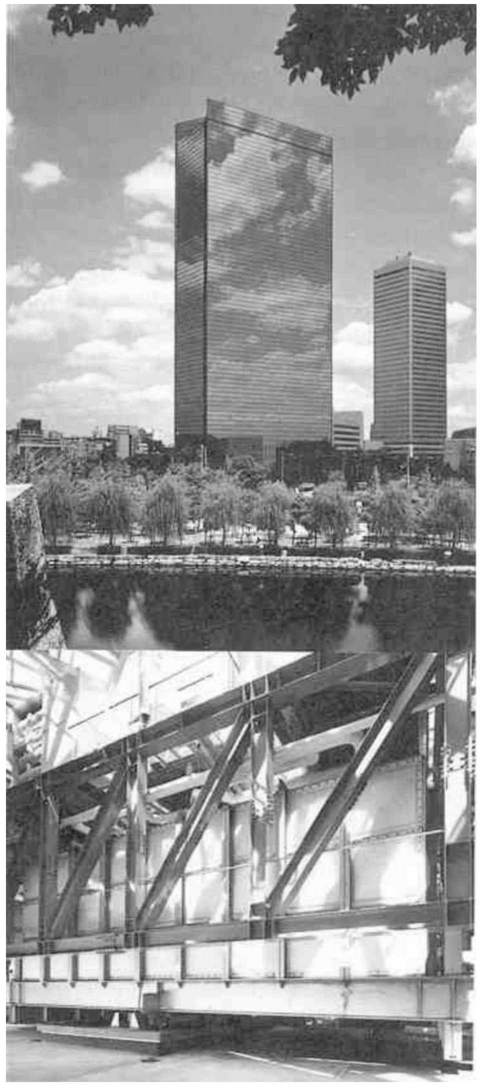


FIGURE 4.11: Ice storage tank—Crystal Tower. (Courtesy of Takemaka Corporation.)

$$m_d[\ddot{u}_d + \ddot{u}] + k_d u_d = -m_d a_g \tag{4.29}$$

$$m\ddot{u} + ku - k_d u_d = -m a_g + p \tag{4.30}$$

where a_g is the absolute ground acceleration and p is the force loading applied to the primary mass.

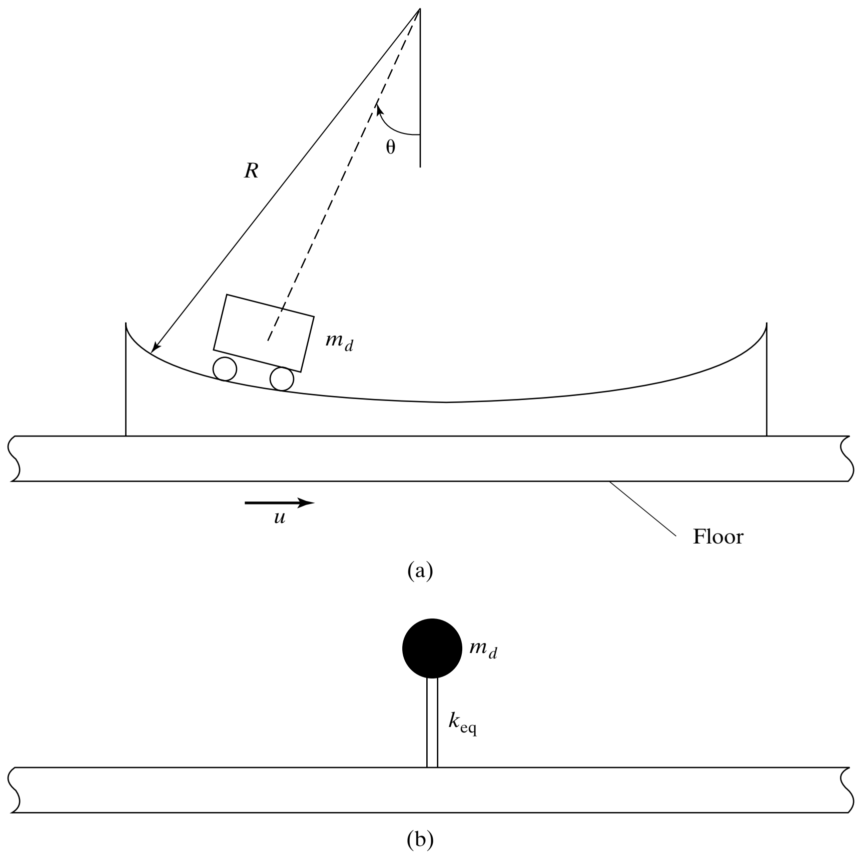


FIGURE 4.12: Rocker pendulum.

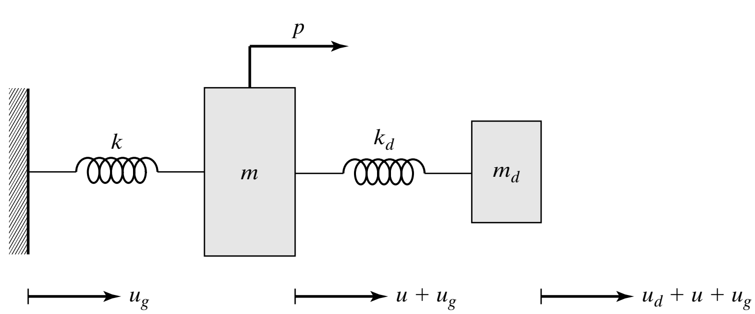


FIGURE 4.13: SDOF system coupled with a TMD.

The excitation is considered to be periodic of frequency Ω,

$$a_g = \hat{a}_g \sin \Omega t \tag{4.31}$$

$$p = \hat{p} \sin \Omega t \tag{4.32}$$

Expressing the response as

$$u = \hat{u} \sin \Omega t \tag{4.33}$$

$$u_d = \hat{u}_d \sin \Omega t \tag{4.34}$$

and substituting for these variables, the equilibrium equations are transformed to

$$[-m_d\Omega^2 + k_d]\hat{u}_d - m_d\Omega^2\hat{u} = -m_d\hat{a}_g \tag{4.35}$$

$$-k_d\hat{u}_d + [-m\Omega^2 + k]\hat{u} = -m\hat{a}_g + \hat{p} \tag{4.36}$$

The solutions for $\hat{u}$ and $\hat{u}_d$ are given by

$$\hat{u} = \frac{\hat{p}}{k}\left(\frac{1-\rho_d^2}{D_1}\right) - \frac{m\hat{a}_g}{k}\left(\frac{1+\overline{m}-\rho_d^2}{D_1}\right) \tag{4.37}$$

$$\hat{u}_d = \frac{\hat{p}}{k_d}\left(\frac{\overline{m}\rho^2}{D_1}\right) - \frac{m\hat{a}_g}{k_d}\left(\frac{\overline{m}}{D_1}\right) \tag{4.38}$$

where

$$D_1 = [1-\rho^2][1-\rho_d^2] - \overline{m}\rho^2 \tag{4.39}$$

and the ρ terms are dimensionless frequency ratios,

$$\rho = \frac{\Omega}{\omega} = \frac{\Omega}{\sqrt{k/m}} \tag{4.40}$$

$$\rho_d = \frac{\Omega}{\omega_d} = \frac{\Omega}{\sqrt{k_d/m_d}} \tag{4.41}$$

Selecting the mass ratio and damper frequency ratio such that

$$1 - \rho_d^2 + \overline{m} = 0 \tag{4.42}$$

reduces the solution to

$$\hat{u} = \frac{\hat{p}}{k} \tag{4.43}$$

$$\hat{u}_d = -\frac{\hat{p}}{k_d}\rho^2 + \frac{m\hat{a}_g}{k_d} \tag{4.44}$$

This choice *isolates* the primary mass from ground motion and reduces the response due to external force to the pseudostatic value, $\hat{p}/k$. A typical range for $\bar{m}$ is 0.01 to 0.1. Then the *optimal* damper frequency is very close to the forcing frequency. The exact relationship follows from Eq. (4.42).

$$\omega_d\big|_{\text{opt}} = \frac{\Omega}{\sqrt{1+\bar{m}}} \tag{4.45}$$

We determine the corresponding damper stiffness with

$$k_d\big|_{\text{opt}} = [\omega_d\big|_{\text{opt}}]^2 m_d = \frac{\Omega^2 m\bar{m}}{1+\bar{m}} \tag{4.46}$$

Finally, substituting for k_d, Eq. (4.44) takes the following form:

$$\hat{u}_d = \frac{1+\bar{m}}{\bar{m}}\left(\left|\frac{\hat{p}}{k}\right| + \left|\frac{\hat{a}_g}{\Omega^2}\right|\right) \tag{4.47}$$

We specify the amount of relative displacement for the damper and determine $\bar{m}$ with Eq. (4.47). Given $\bar{m}$ and Ω, the stiffness is found using Eq. (4.46). It should be noted that this stiffness applies for a *particular* forcing frequency. Once the mass damper properties are defined, Eqs. (4.37) and (4.38) can be used to determine the response for a *different* forcing frequency. The primary mass will move under ground motion excitation in this case.

4.4.2 Undamped Structure: Damped TMD

The next level of complexity has damping included in the mass damper, as shown in Figure 4.14. The equations of motion for this case are

$$m_d\ddot{u}_d + c_d\dot{u}_d + k_d u_d + m_d\ddot{u} = -m_d a_g \tag{4.48}$$

$$m\ddot{u} + ku - c_d\dot{u}_d - k_d u_d = -ma_g + p \tag{4.49}$$

The inclusion of the damping terms in Eqs. (4.48) and (4.49) produces a phase shift between the periodic excitation and the response. It is convenient to work initially with the solution expressed in terms of complex quantities. We express the excitation as

$$a_g = \hat{a}_g e^{i\Omega t} \tag{4.50}$$

$$p = \hat{p} e^{i\Omega t} \tag{4.51}$$

where $\hat{a}_g$ and $\hat{p}$ are real quantities. The response is taken as

$$u = \bar{u} e^{i\Omega t} \tag{4.52}$$

$$u_d = \bar{u}_d e^{i\Omega t} \tag{4.53}$$

where the response amplitudes, $\bar{u}$ and $\bar{u}_d$, are considered to be complex quantities. The real and imaginary parts of a_g correspond to cosine and sinusoidal input. Then the corresponding solution is given by either the real (for cosine) or imaginary (for sine) parts of u and u_d. Substituting Eqs. (4.52) and (4.53) in the set of governing equations and cancelling $e^{i\Omega t}$ from both sides results in

$$[-m_d\Omega^2 + ic_d\Omega + k_d]\bar{u}_d - m_d\Omega^2\bar{u} = -m_d\hat{a}_g \tag{4.54}$$

$$-[ic_d\Omega + k_d]\bar{u}_d + [-m\Omega^2 + k]\bar{u} = -m\hat{a}_g + \hat{p} \tag{4.55}$$

The solution of the governing equations is

$$\bar{u} = \frac{\hat{p}}{kD_2}[f^2 - \rho^2 + i2\xi_d\rho f] - \frac{\hat{a}_g m}{kD_2}[(1+\bar{m})f^2 - \rho^2 + i2\xi_d\rho f(1+\bar{m})] \tag{4.56}$$

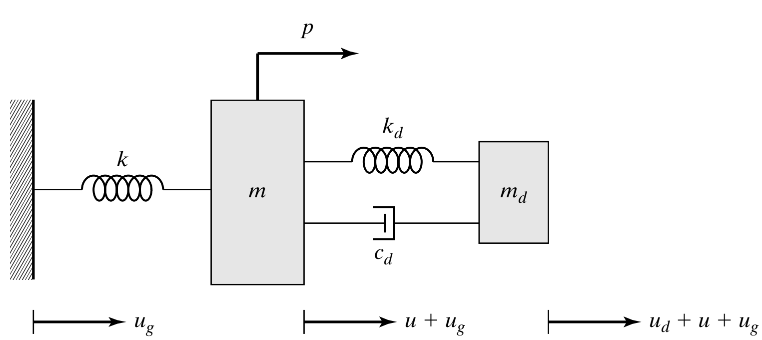


FIGURE 4.14: Undamped SDOF system coupled with a damped TMD system.

$$\bar{u}_d = \frac{\hat{p}\rho^2}{kD_2} - \frac{\hat{a}_g m}{kD_2} \tag{4.57}$$

where

$$D_2 = [1-\rho^2][f^2-\rho^2] - \bar{m}\rho^2 f^2 + i2\xi_d\rho f[1-\rho^2(1+\bar{m})] \tag{4.58}$$

$$f = \frac{\omega_d}{\omega} \tag{4.59}$$

and ρ was defined earlier as the ratio of Ω to ω [see Eq. (4.40)].

Converting the complex solutions to polar form leads to the following expressions:

$$\bar{u} = \frac{\hat{p}}{k}H_1 e^{i\delta_1} - \frac{\hat{a}_g m}{k}H_2 e^{i\delta_2} \tag{4.60}$$

$$\bar{u}_d = \frac{\hat{p}}{k}H_3 e^{-i\delta_3} - \frac{\hat{a}_g m}{k}H_4 e^{-i\delta_3} \tag{4.61}$$

where the H factors define the amplification of the pseudo-static responses, and the δ's are the phase angles between the response and the excitation. The various H and δ terms are as follows:

$$H_1 = \frac{\sqrt{[f^2-\rho^2]^2 + [2\xi_d\rho f]^2}}{|D_2|} \tag{4.62}$$

$$H_2 = \frac{\sqrt{[(1+\bar{m})f^2-\rho^2]^2 + [2\xi_d\rho f(1+\bar{m})]^2}}{|D_2|} \tag{4.63}$$

$$H_3 = \frac{\rho^2}{|D_2|} \tag{4.64}$$

$$H_4 = \frac{1}{|D_2|} \tag{4.65}$$

$$|D_2| = \sqrt{([1-\rho^2][f^2-\rho^2] - \bar{m}\rho^2 f^2)^2 + (2\xi_d\rho f[1-\rho^2(1+\bar{m})])^2} \tag{4.66}$$

Also,

$$\delta_1 = \alpha_1 - \delta_3 \tag{4.67}$$

$$\delta_2 = \alpha_2 - \delta_3 \tag{4.68}$$

$$\tan\delta_3 = \frac{2\xi_d\rho f[1-\rho^2(1+\bar{m})]}{[1-\rho^2][f^2-\rho^2]-\bar{m}\rho^2 f^2} \tag{4.69}$$

$$\tan\alpha_1 = \frac{2\xi_d\rho f}{f^2-\rho^2} \tag{4.70}$$

$$\tan\alpha_2 = \frac{2\xi_d\rho f(1+\bar{m})}{(1+\bar{m})f^2-\rho^2} \tag{4.71}$$

For most applications, the mass ratio is less than about 0.05. Then the amplification factors for external loading (H_1) and ground motion (H_2) are essentially equal. A similar conclusion applies for the phase shift. In what follows, the solution corresponding to ground motion is examined and the optimal values of the damper properties for this loading condition are established. An in-depth treatment of the external forcing case is contained in Den Hartog's text (Den Hartog, 1940).

Figure 4.15 shows the variation of H_2 with forcing frequency for specific values of damper mass $\bar{m}$ and frequency ratio f, and various values of the damper damping ratio, ξ_d. When $\xi_d = 0$, there are two peaks with infinite amplitude located on each side of $\rho = 1$. As ξ_d is increased, the peaks approach each other and then merge into a single peak located at $\rho \approx 1$. The behavior of the amplitudes suggests that there is an optimal value of ξ_d for a given damper configuration (m_d and k_d or, equivalently, $\bar{m}$ and f). Another key observation is that all the curves pass through two common points, P and Q. Since these curves correspond to different values of ξ_d, the location of P and Q must depend only on $\bar{m}$ and f.

Proceeding with this line of reasoning, the expression for H_2 can be written as

$$H_2 = \sqrt{\frac{a_1^2+\xi_d^2 a_2^2}{a_3^2+\xi_d^2 a_4^2}} = \frac{a_2}{a_4}\sqrt{\frac{a_1^2/a_2^2+\xi_d^2}{a_3^2/a_4^2+\xi_d^2}} \tag{4.72}$$

where the a terms are functions of $\bar{m}$, ρ, and f. Then for H_2 to be independent of ξ_d, the following condition must be satisfied:

$$\left|\frac{a_1}{a_2}\right| = \left|\frac{a_3}{a_4}\right| \tag{4.73}$$

The corresponding values for H_2 are

$$H_2\big|_{P,\,Q} = \left|\frac{a_2}{a_4}\right| \tag{4.74}$$

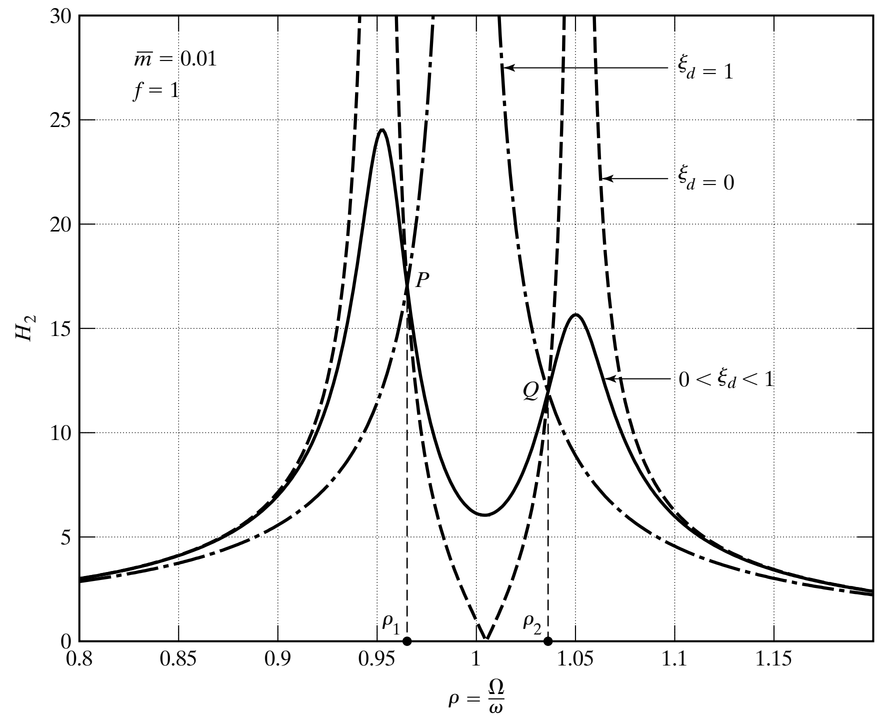


FIGURE 4.15: Plot of H_2 versus ρ.

Substituting for the *a* terms in Eq. (4.73), we obtain a quadratic equation for ρ^2:

$$\rho^4 - \left[(1+\bar{m})f^2 + \frac{1+0.5\bar{m}}{1+\bar{m}}\right]\rho^2 + f^2 = 0 \tag{4.75}$$

The two positive roots ρ_1 and ρ_2 are the frequency ratios corresponding to points P and Q. Similarly, Eq. (4.74) expands to

$$H_2\big|_{P,Q} = \frac{1+\bar{m}}{\left|1-\rho_{1,2}^2(1+\bar{m})\right|} \tag{4.76}$$

Figure 4.15 shows different values for H_2 at points P and Q. For *optimal* behavior, we want to minimize the maximum amplitude. As a first step, we require the values of H_2 for ρ_1 and ρ_2 to be equal. This produces a distribution that is symmetrical about $\rho^2 = 1/(1+\bar{m})$, as illustrated in Figure 4.16. Then, by increasing the damping ratio, ξ_d, we can lower the peak amplitudes until the peaks coincide with points P and Q. This state represents the **optimal** performance of the TMD system. A further increase in ξ_d causes the peaks to merge and the amplitude to increase beyond the optimal value.

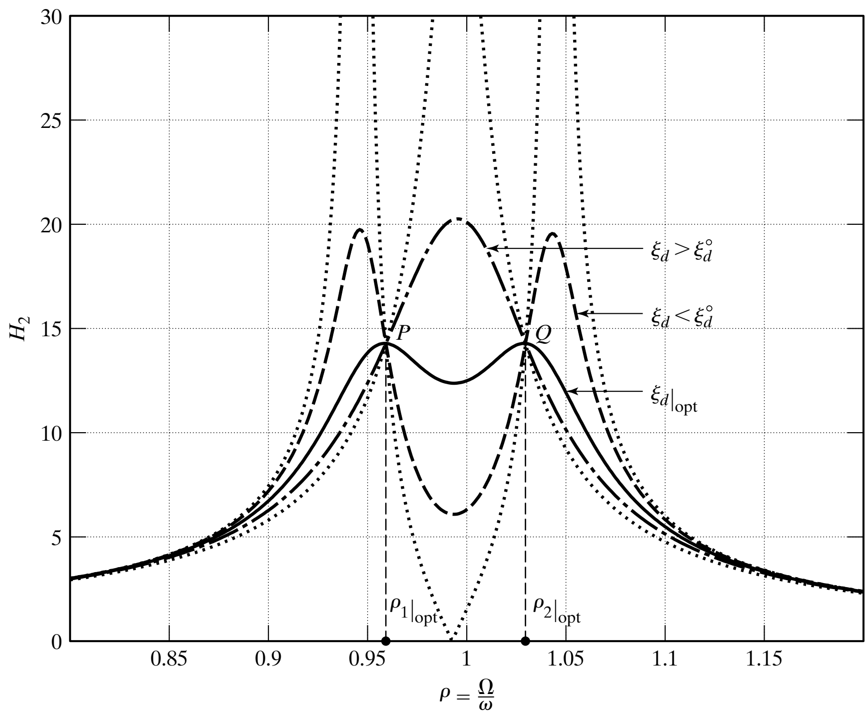


FIGURE 4.16: Plot of H_2 versus ρ for f_{opt}.

Requiring the amplitudes to be equal at P and Q is equivalent to the following condition on the roots:

$$\left|1 - \rho_1^2(1 + \bar{m})\right| = \left|1 - \rho_2^2(1 + \bar{m})\right| \tag{4.77}$$

Then, substituting for ρ_1 and ρ_2 using Eq. (4.75), we obtain a relation between the optimal tuning frequency and the mass ratio:

$$f_{opt} = \frac{\sqrt{1 - 0.5\bar{m}}}{1 + \bar{m}} \tag{4.78}$$

$$\omega_d|_{opt} = f_{opt}\omega \tag{4.79}$$

The corresponding roots and optimal amplification factors are

$$\rho_{1,2}\big|_{opt} = \sqrt{\frac{1 \pm \sqrt{0.5\bar{m}}}{1+\bar{m}}} \tag{4.80}$$

$$H_2\big|_{opt} = \frac{1+\bar{m}}{\sqrt{0.5\bar{m}}} \tag{4.81}$$

The expression for the optimal damping at the optimal tuning frequency is

$$\xi_d\big|_{opt} = \sqrt{\frac{\bar{m}(3-\sqrt{0.5\bar{m}})}{8(1+\bar{m})(1-0.5\bar{m})}} \tag{4.82}$$

Figures 4.17 through 4.20 show the variation of the *optimal* parameters with the mass ratio, $\bar{m}$.

The response of the damper is defined by Eq. (4.61). Specializing this equation for the optimal conditions leads to the plot of amplification versus mass ratio contained in Figure 4.21. A comparison of the damper motion with respect to the motion of the primary mass for optimal conditions is shown in Figure 4.22.

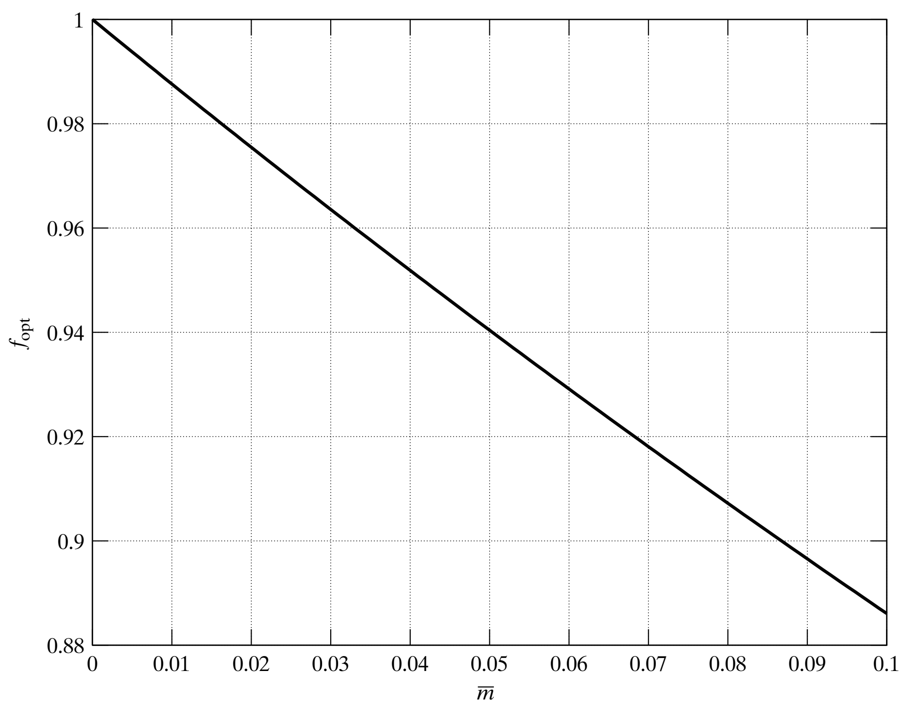


FIGURE 4.17: Optimum tuning frequency ratio, f_{opt}.

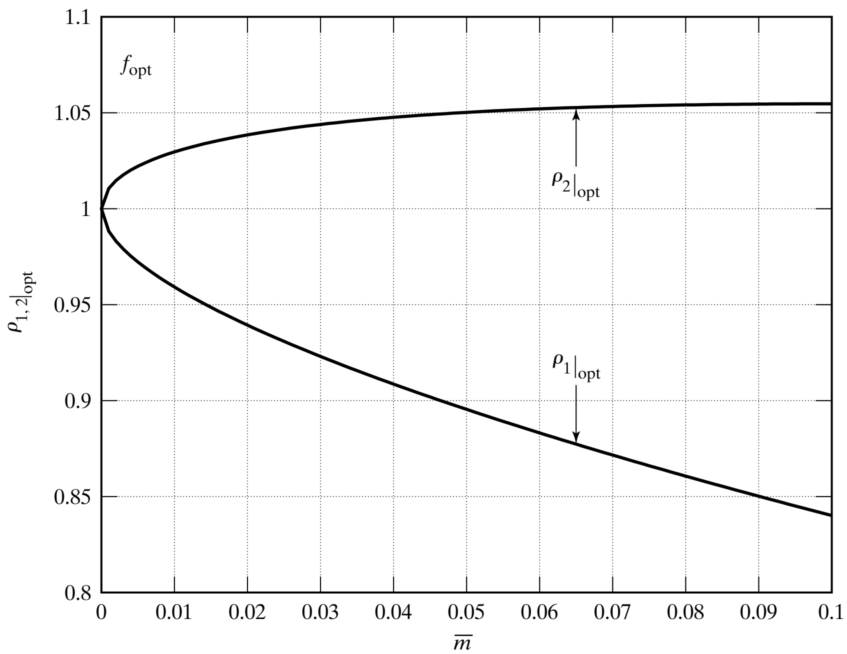


FIGURE 4.18: Input frequency ratios at which the response is independent of damping.

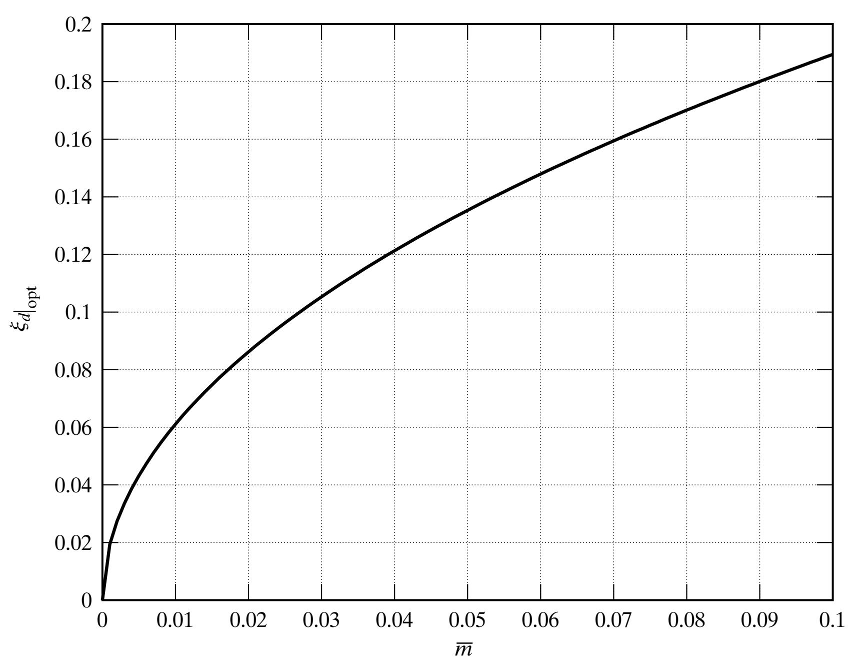


FIGURE 4.19: Optimal damping ratio for TMD.

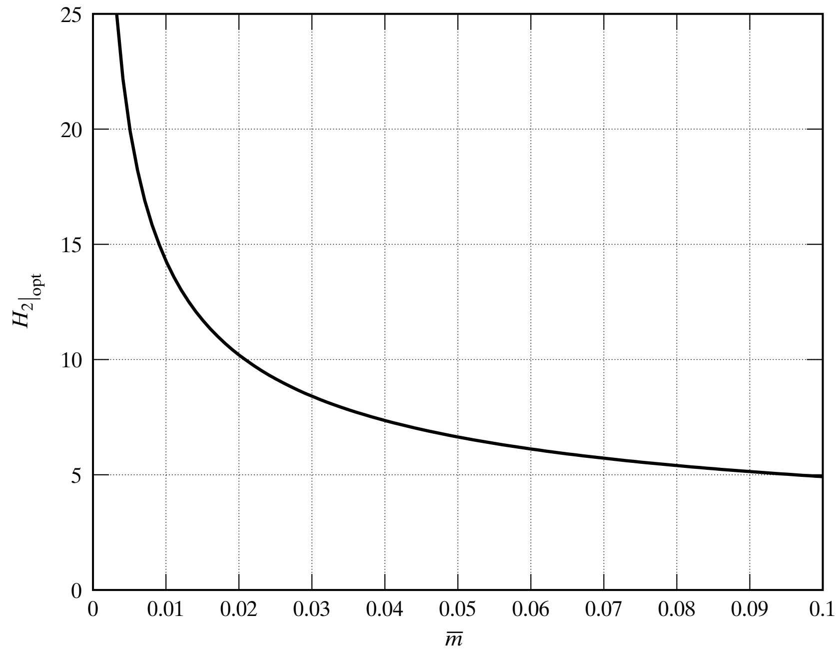


FIGURE 4.20: Maximum dynamic amplification factor for SDOF system (optimal tuning and damping).

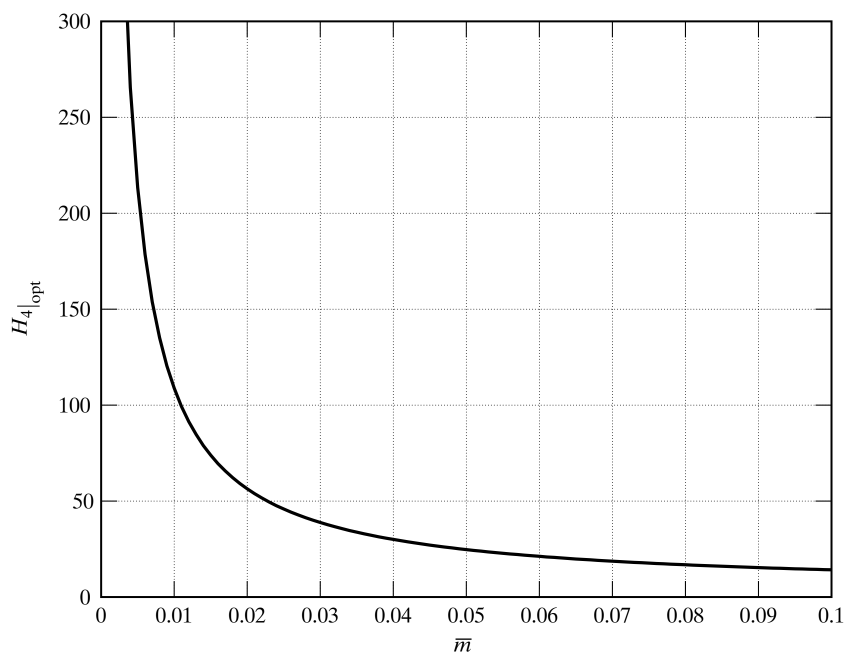


FIGURE 4.21: Maximum dynamic amplification factor for TMD.

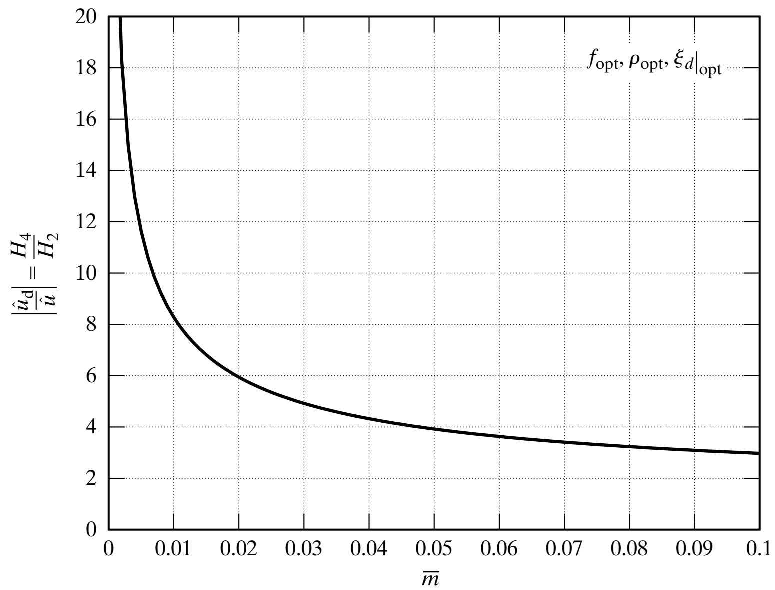


FIGURE 4.22: Ratio of maximum TMD amplitude to maximum system amplitude.

Lastly, response curves for a typical mass ratio, $\overline{m} = 0.01$, and optimal tuning are plotted in Figure 4.23 and Figure 4.24. The response for no damper is also plotted in Figure 4.23. We observe that the effect of the damper is to limit the motion in a frequency range centered on the natural frequency of the primary mass and extending about 0.15ω. Outside of this range, the motion is not significantly influenced by the damper.

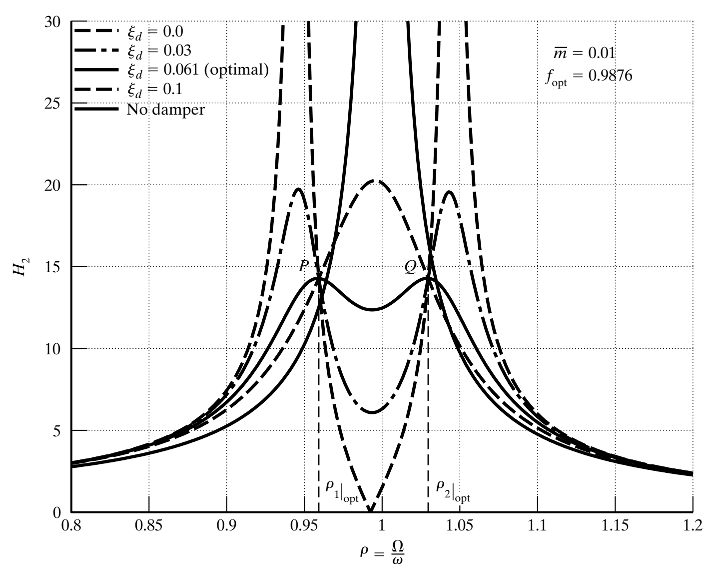


FIGURE 4.23: Response curves for amplitude of system with optimally tuned TMD.

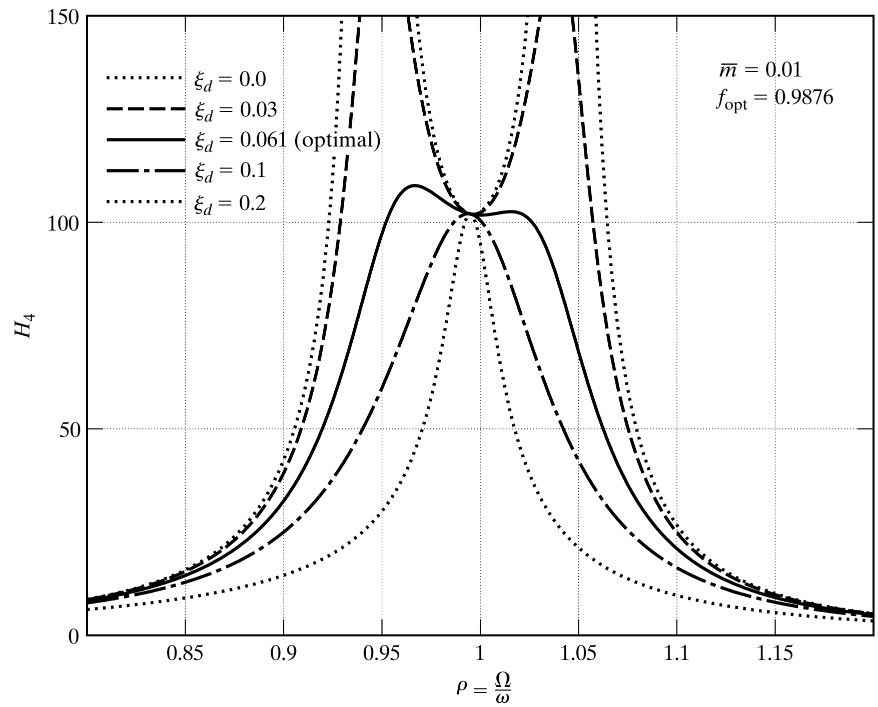


FIGURE 4.24: Response curves for amplitude of optimally tuned TMD.

The maximum amplification for a damped SDOF system without a TMD, undergoing harmonic excitation, is given by Eq. (1.32):

$$H = \frac{1}{2\xi\sqrt{1-\xi^2}} \tag{4.83}$$

Since ξ is small, a reasonable approximation is

$$H \approx \frac{1}{2\xi} \tag{4.84}$$

Expressing the optimal H_2 in a similar form provides a measure of the equivalent damping ratio ξ_e for the primary mass:

$$\xi_e = \frac{1}{2H_2|_{\text{opt}}} \tag{4.85}$$

Figure 4.25 shows the variation of ξ_e with the mass ratio. A mass ratio of 0.02 is equivalent to about 5% damping in the primary system.

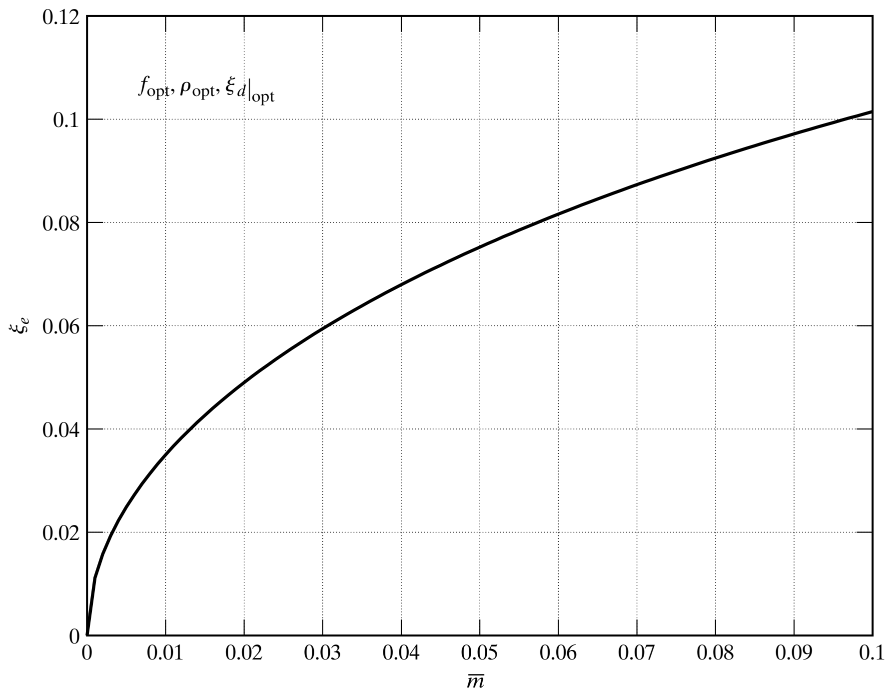


FIGURE 4.25: Equivalent damping ratio for optimally tuned TMD.

The design of a TMD involves the following steps:

- Establish the allowable values of displacement of the primary mass and the TMD for the design loading. This data provides the *design* values for $H_2|_{opt}$ and $H_4|_{opt}$.
- Determine the mass ratios required to satisfy these motion constraints from Figure 4.20 and Figure 4.21. Select the largest value of $\bar{m}$.
- Determine f_{opt} form Figure 4.17.
- Compute ω_d:

$$\omega_d = f_{opt}\omega \tag{4.86}$$

- Compute k_d:

$$k_d = m_d\omega_d^2 = \bar{m}kf_{opt}^2 \tag{4.87}$$

- Determine $\xi_d|_{opt}$ from Figure 4.19.
- Compute c_d:

$$c_d = 2\xi_d|_{opt}\omega_d m_d = \bar{m}f_{opt}\left[2\xi_d|_{opt}\omega m\right] \tag{4.88}$$

Example 4.2: Design of a TMD for an undamped SDOF system

Consider the following motion constraints:

$$H_2|_{\text{opt}} < 7 \tag{1}$$

$$\frac{H_4}{H_2|_{\text{opt}}} < 6 \tag{2}$$

Constraint Eq. (1) requires $\bar{m} \geq 0.05$. For constraint Eq. (2), we need to take $\bar{m} \geq 0.02$. Therefore, $\bar{m} \geq 0.05$ controls the design. The relevant parameters are

$$\bar{m} = 0.05 \qquad f_{\text{opt}} = 0.94 \qquad \xi_d|_{\text{opt}} = 0.135$$

Then

$$m_d = 0.05m \qquad \omega_d = 0.94\omega \qquad k_d = \bar{m} f_{\text{opt}}^2 k = 0.044k$$

4.4.3 Damped Structure: Damped TMD

All real systems contain some damping. Although an absorber is likely to be added only to a lightly damped system, assessing the effect of damping in the real system on the optimal tuning of the absorber is an important design consideration.

The main system in Figure 4.26 consists of the mass m, spring stiffness k, and viscous damping c. The TMD system has mass m_d, stiffness k_d, and viscous damping c_d. Considering the system to be subjected to both external forcing and ground excitation, the equations of motion are

$$m_d \ddot{u}_d + c_d \dot{u}_d + k_d u_d + m_d \ddot{u} = -m_d a_g \tag{4.89}$$

$$m\ddot{u} + c\dot{u} + ku - c_d \dot{u}_d - k_d u_d = -m a_g + p \tag{4.90}$$

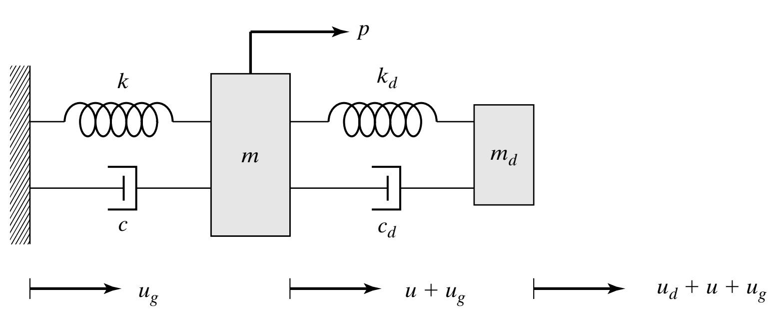


FIGURE 4.26: Damped SDOF system coupled with a damped TMD system.

Proceeding in the same way as for the undamped case, the solution due to periodic excitation (both p and u_g) is expressed in polar form:

$$\bar{u} = \frac{\hat{p}}{k} H_5 \, e^{i\delta_5} - \frac{\hat{a}_g m}{k} H_6 \, e^{i\delta_6} \tag{4.91}$$

$$\bar{u}_d = \frac{\hat{p}}{k} H_7 \, e^{-i\delta_7} - \frac{\hat{a}_g m}{k} H_8 \, e^{i\delta_8} \tag{4.92}$$

The various H and δ terms are defined as follows:

$$H_5 = \frac{\sqrt{[f^2 - \rho^2]^2 + [2\xi_d \rho f]^2}}{|D_3|} \tag{4.93}$$

$$H_6 = \frac{\sqrt{[(1+\bar{m})f^2 - \rho^2]^2 + [2\xi_d \rho f(1+\bar{m})]^2}}{|D_3|} \tag{4.94}$$

$$H_7 = \frac{\rho^2}{|D_3|} \tag{4.95}$$

$$H_8 = \frac{\sqrt{1 + [2\xi\rho]^2}}{|D_3|} \tag{4.96}$$

$$\begin{aligned} |D_3| = \{ & [-f^2\rho^2\bar{m} + (1-\rho^2)(f^2-\rho^2) - 4\xi\xi_d f\rho^2]^2 \\ & + 4[\xi\rho(f^2-\rho^2) + \xi_d f\rho(1-\rho^2(1+\bar{m}))^2] \} \end{aligned} \tag{4.97}$$

$$\delta_5 = \alpha_1 - \delta_7 \tag{4.98}$$

$$\delta_6 = \alpha_2 - \delta_7 \tag{4.99}$$

$$\delta_8 = \alpha_3 - \delta_7 \tag{4.100}$$

$$\tan\delta_7 = 2\frac{\xi\rho(f^2-\rho^2) + \xi_d f\rho(1-\rho^2(1+\bar{m}))}{-f^2\rho^2\bar{m} + (1-\rho^2)(f^2-\rho^2) - 4\xi\xi_d f\rho^2} \tag{4.101}$$

$$\tan\alpha_3 = 2\xi\rho \tag{4.102}$$

The α_1 and α_2 terms are defined by Eqs. (4.70) and (4.71).

In what follows, the case of an external force applied to the primary mass is considered. Since $|D_3|$ involves ξ, we cannot establish analytical expressions for the optimal tuning frequency and optimal damping ratio in terms of the mass ratio. In this case, these parameters also depend on ξ. Numerical simulations can be applied to evaluate H_5 and H_7 for a range of ρ, given the values for $\overline{m}$, ξ, f, and ξ_d. Starting with specific values for $\overline{m}$ and ξ, plots of H_5 versus ρ can be generated for a range of f and ξ_d. Each $H_5 - \rho$ plot has a peak value of H_5. The particular combination of f and ξ_d that corresponds to the *lowest* peak value of H_5 is taken as the *optimal* state. Repeating this process for different values of $\overline{m}$ and ξ produces the behavioral data needed to design the damper system.

Figure 4.27 shows the variation of the maximum value of H_5 for the optimal state. The corresponding response of the damper is plotted in Figure 4.28. Adding damping to the primary mass has an appreciable effect for small $\overline{m}$. Noting Eqs. (4.91) and (4.92), the ratio of damper displacement to primary mass displacement is given by

$$\frac{|\hat{u}_d|}{|\hat{u}|} = \frac{H_7}{H_5} = \frac{\rho^2}{\sqrt{[f^2 - \rho^2]^2 + [2\xi_d \rho f]^2}} \tag{4.103}$$

Since ξ is small, this ratio is essentially independent of ξ. Figure 4.29 confirms this statement. The optimal values of the frequency and damping ratios generated through simulation are plotted in Figures 4.30 and 4.31. Lastly, using Eq. (4.85), $H_5|_{\text{opt}}$ can be converted to an equivalent damping ratio for the primary system.

$$\xi_e = \frac{1}{2H_5|_{\text{opt}}} \tag{4.104}$$

Figure 4.32 shows the variation of ξ_e with $\overline{m}$ and ξ.

Tsai and Lin (1993) suggest equations for the optimal tuning parameters f and ξ_d determined by curve fitting schemes. The equations are listed next for completeness.

$$\begin{aligned} f = {} & \left(\frac{\sqrt{1 - 0.5\overline{m}}}{1 + \overline{m}} + \sqrt{1 - 2\xi^2} - 1 \right) \\ & - [2.375 - 1.034\sqrt{\overline{m}} - 0.426\overline{m}]\xi\sqrt{\overline{m}} \\ & - (3.730 - 16.903\sqrt{\overline{m}} + 20.496\overline{m})\xi^2\sqrt{\overline{m}} \end{aligned} \tag{4.105}$$

$$\begin{aligned} \xi_d = {} & \sqrt{\frac{3\overline{m}}{8(1 + \overline{m})(1 - 0.5\overline{m})}} + (0.151\xi - 0.170\xi^2) \\ & + (0.163\xi + 4.980\xi^2)\overline{m} \end{aligned} \tag{4.106}$$

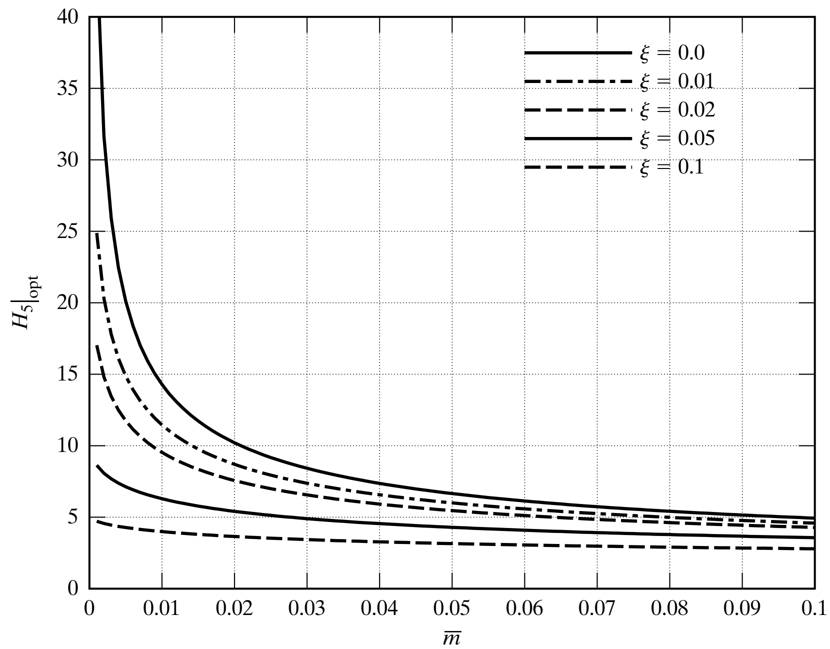


FIGURE 4.27: Maximum dynamic amplification factor for damped SDOF system.

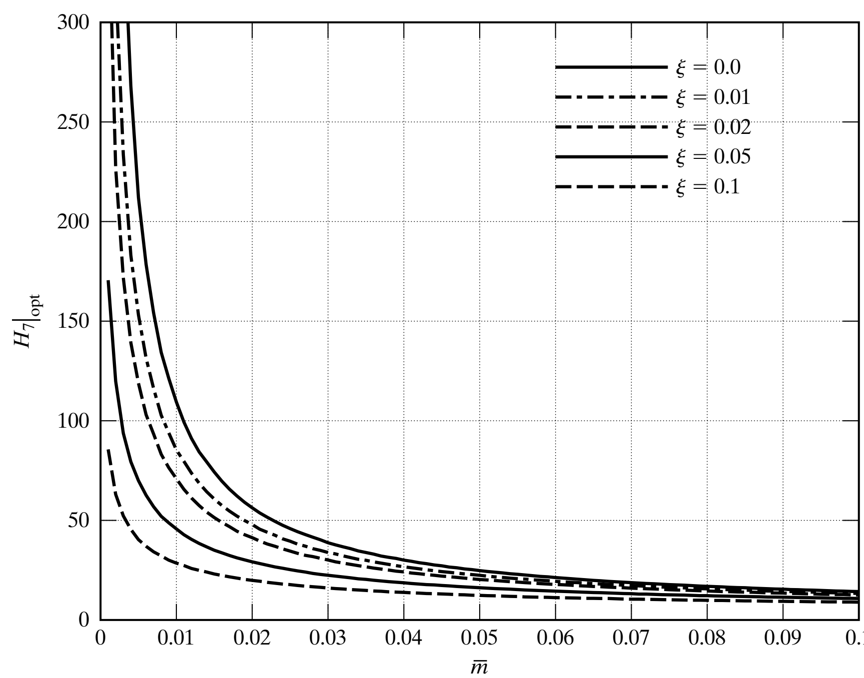


FIGURE 4.28: Maximum dynamic amplification factor for TMD.

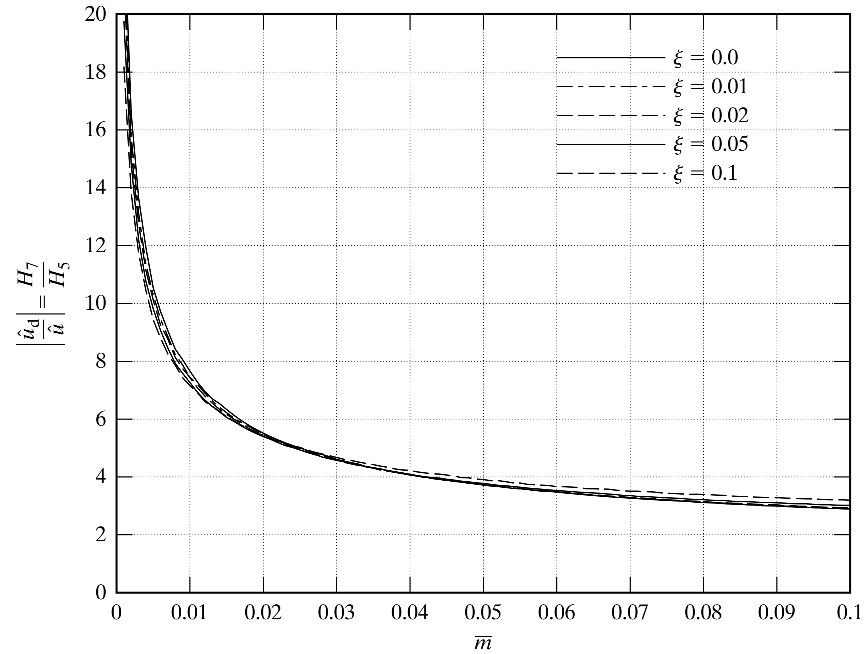


FIGURE 4.29: Ratio of maximum TMD amplitude to maximum system amplitude.

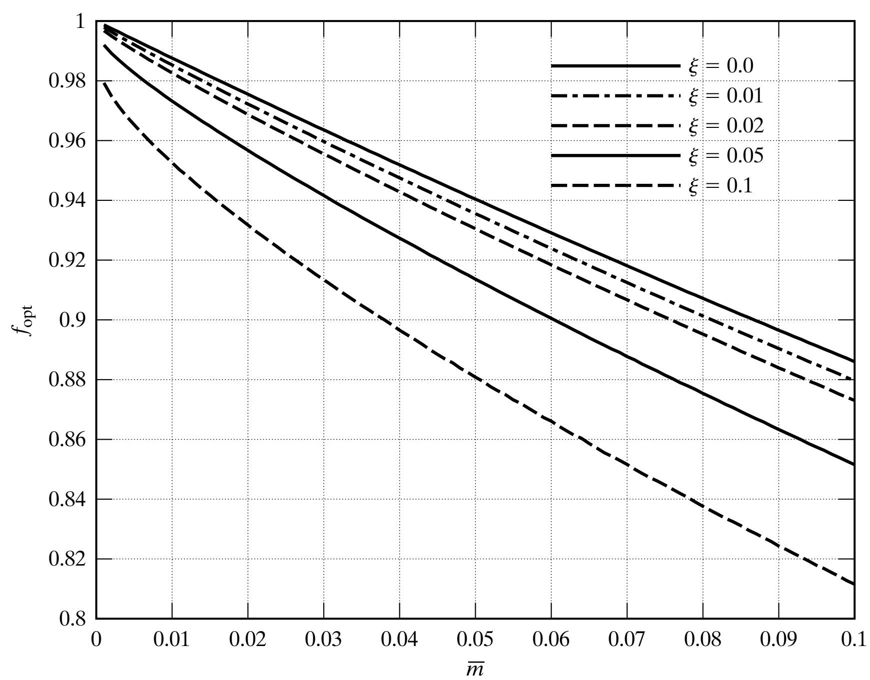


FIGURE 4.30: Optimum tuning frequency ratio for TMD, f_{opt}.

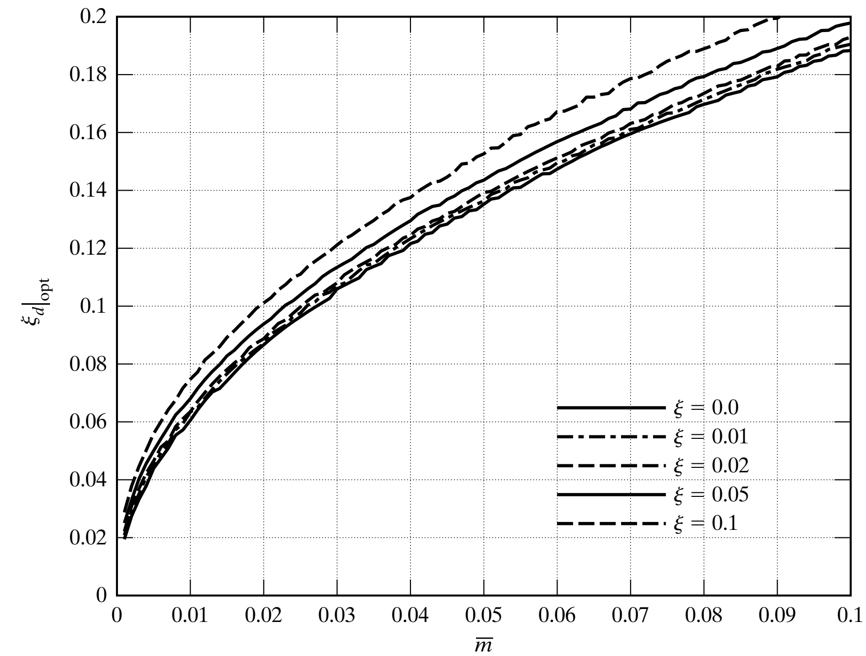


FIGURE 4.31: Optimal damping ratio for TMD.

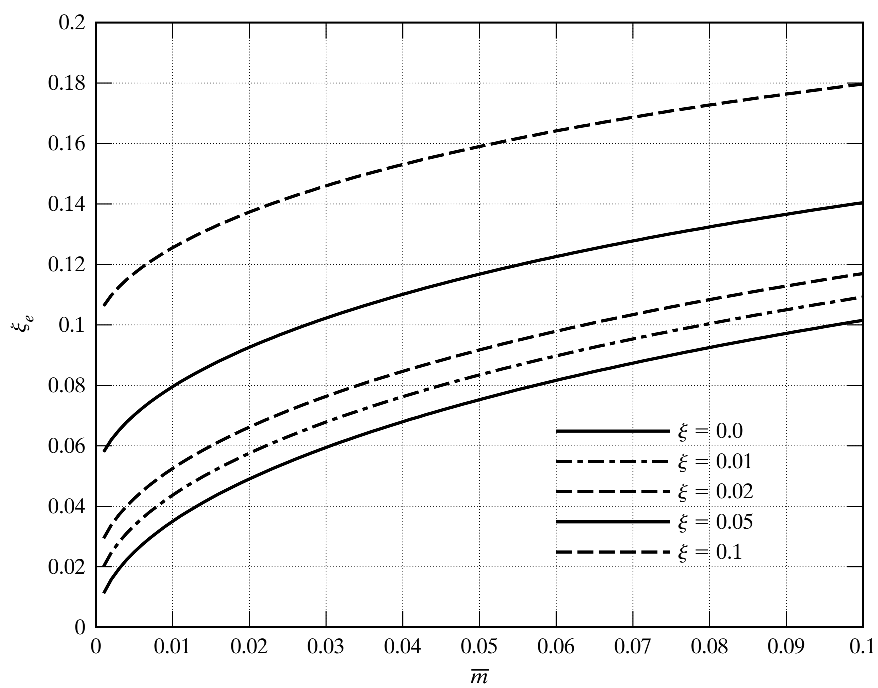


FIGURE 4.32: Equivalent damping ratio for optimally tuned TMD.

Example 4.3: Design of a TMD for a damped SDOF system

Example 4.2 is reworked here, allowing for 2% damping in the primary system. The same design motion constraints are considered:

$$H_5\big|_{\text{opt}} < 7 \tag{1}$$

$$\frac{H_7}{H_5\big|_{\text{opt}}} < 6 \tag{2}$$

Using Figure 4.27, the required mass ratio for $\xi = 0.02$ is $\bar{m} \approx 0.03$. The other optimal values are $f_{\text{opt}} = 0.965$ and $\xi_d\big|_{\text{opt}} = 0.105$. Then

$$m_d = 0.03m \qquad \omega_d = 0.955\omega \qquad k_d = \bar{m}f_{\text{opt}}^2\ k = 0.027k$$

In this case, there is a significant reduction in the damper mass required for this set of motion constraints. The choice between including damping in the primary system versus incorporating a tuned mass damper depends on the relative costs and reliability of the two alternatives, and the nature of the structural problem. A TMD system is generally more appropriate for upgrading an existing structure where access to the structural elements is difficult.

4.5 CASE STUDIES: SDOF SYSTEMS

Figures 4.33 to 4.44 show the time history responses for two SDOF systems with periods of 0.49 s and 5.35 s, respectively under harmonic (at resonance conditions), El Centro, and Taft ground excitations. All examples have a system damping ratio of 2% and an optimally tuned TMD with a mass ratio of 1%. The excitation magnitudes have been scaled so that the peak amplitude of the response of the system without the TMD is unity. The plots show the response of the system without the TMD (the dotted line) as well as the response of the system with the TMD (the solid line). Figures showing the time history of the relative displacement of the TMD with respect to the system are also presented. Significant reduction in the response of the primary system under harmonic excitation is observed. However, optimally tuned mass dampers are relatively ineffective under seismic excitation, and in some cases produce a negative effect (i.e., they amplify the response slightly). This poor performance is attributed to the ineffectiveness of tuned mass dampers for impulsive loadings as well as their inability to reach a resonant condition and therefore dissipate energy under random excitation. These results are in close agreement with the data presented by Kaynia et al. (1981).

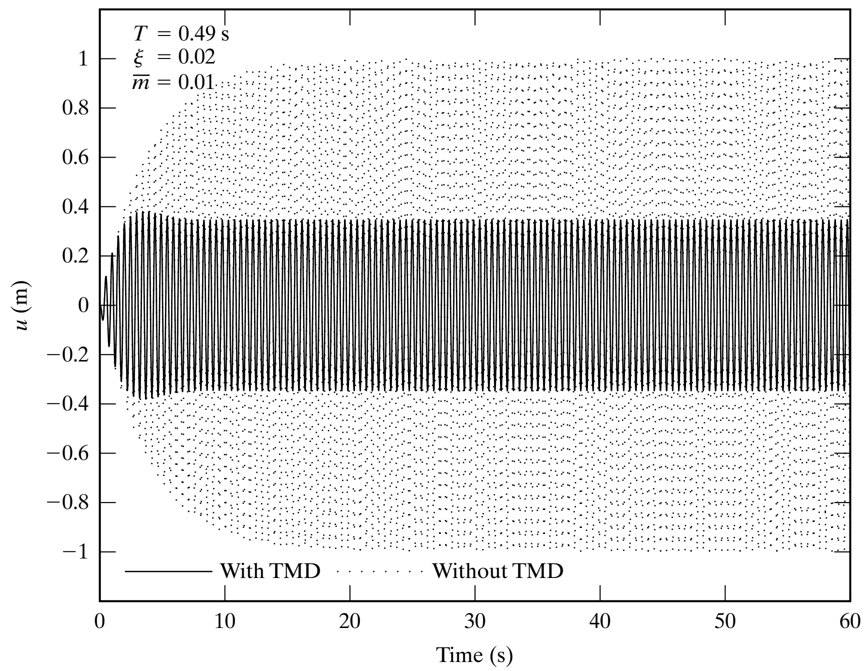


FIGURE 4.33: Response of SDOF to harmonic excitation.

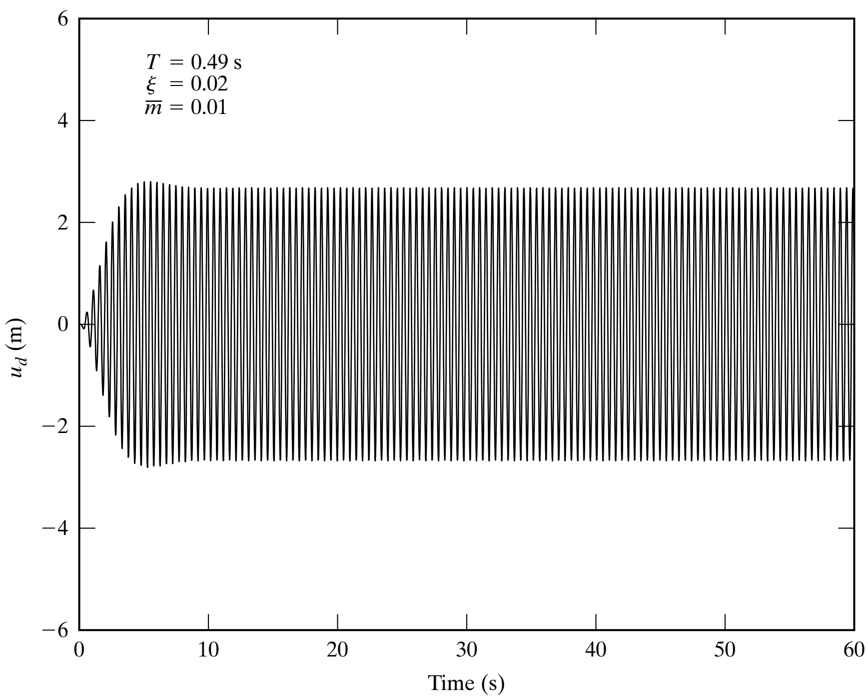


FIGURE 4.34: Relative displacement of TMD under harmonic excitation.

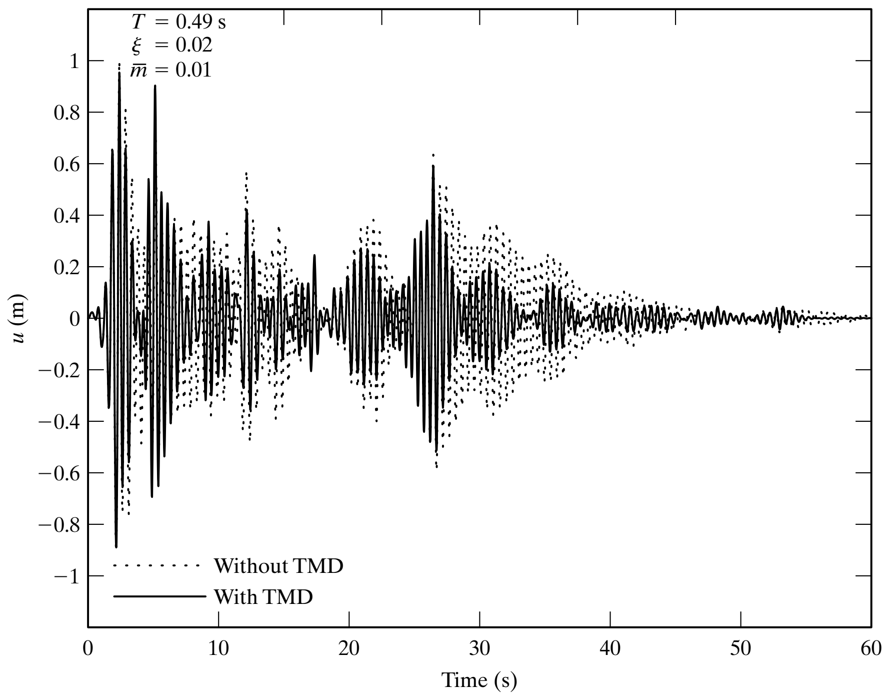


FIGURE 4.35: Response of SDOF to El Centro excitation.

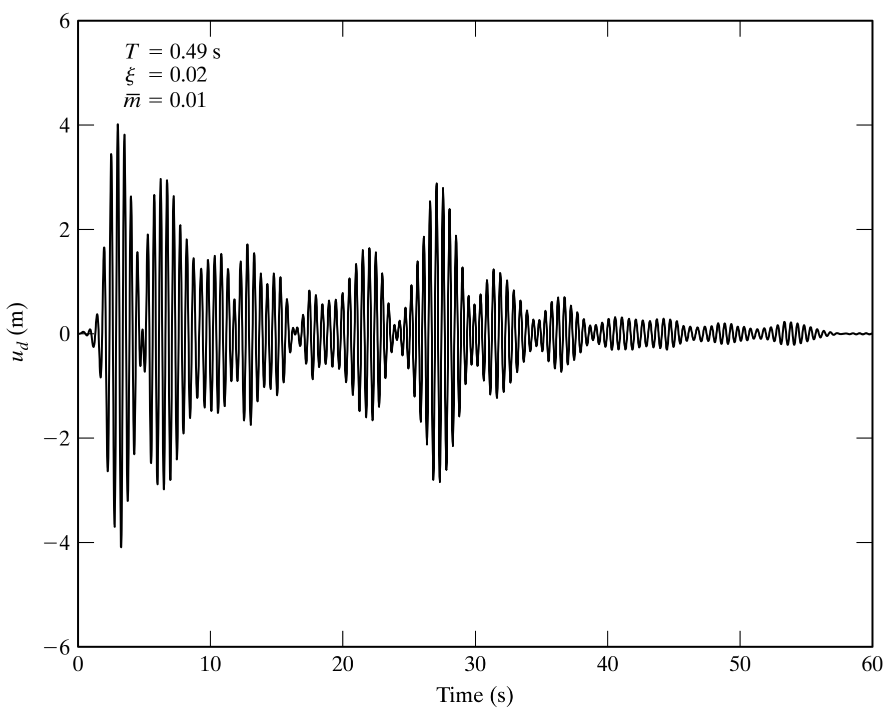


FIGURE 4.36: Relative displacement of TMD under El Centro excitation.

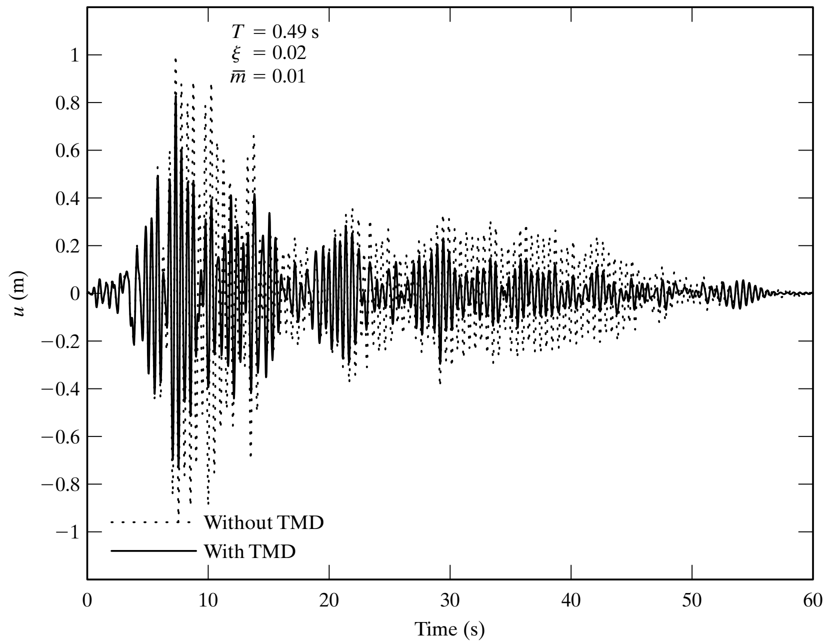


FIGURE 4.37: Response of SDOF to Taft excitation.

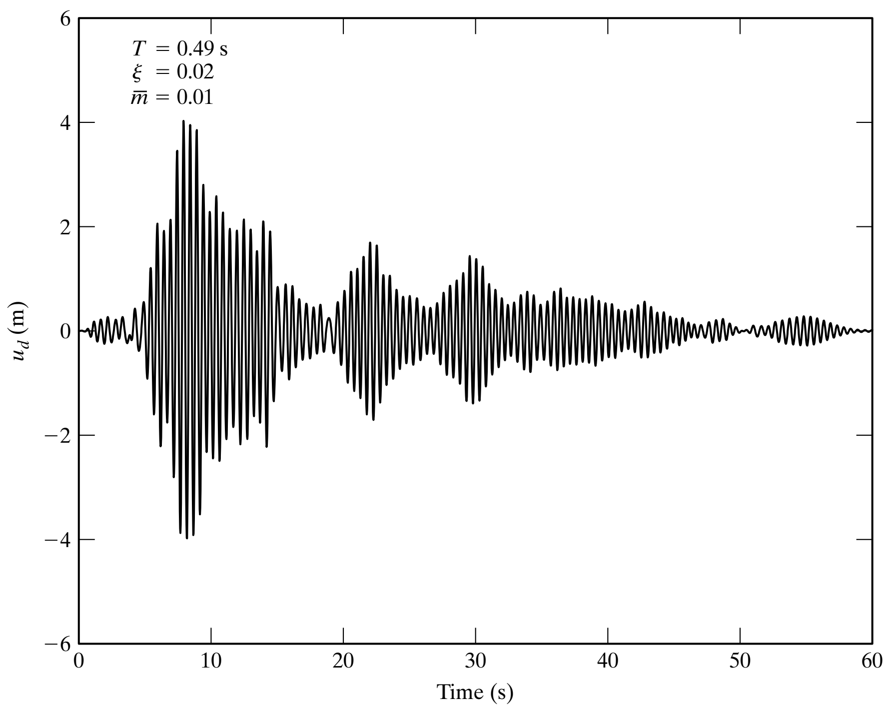


FIGURE 4.38: Relative displacement of TMD under Taft excitation.

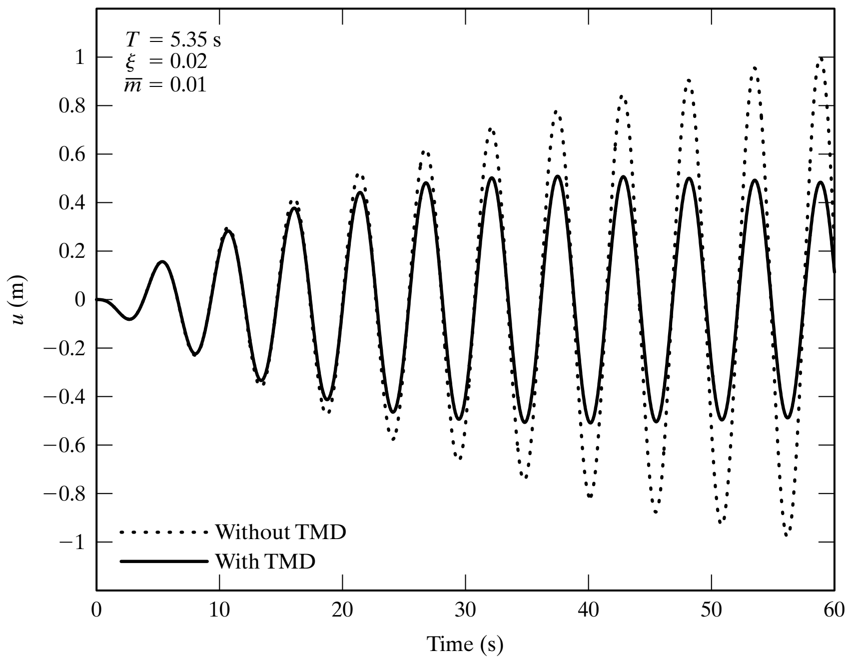


FIGURE 4.39: Response of SDOF to harmonic excitation.

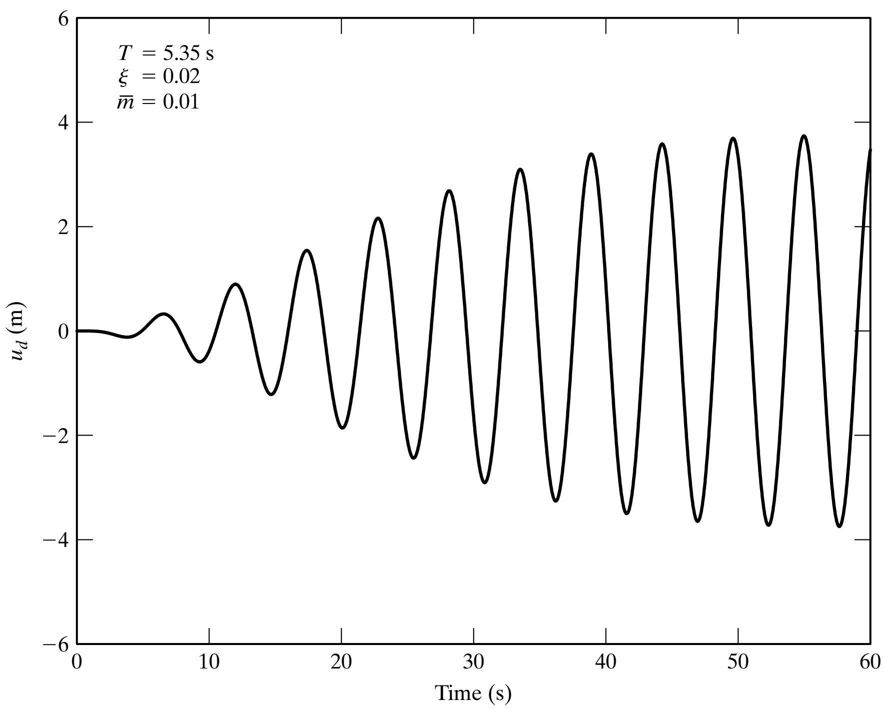


FIGURE 4.40: Relative displacement of TMD under harmonic excitation.

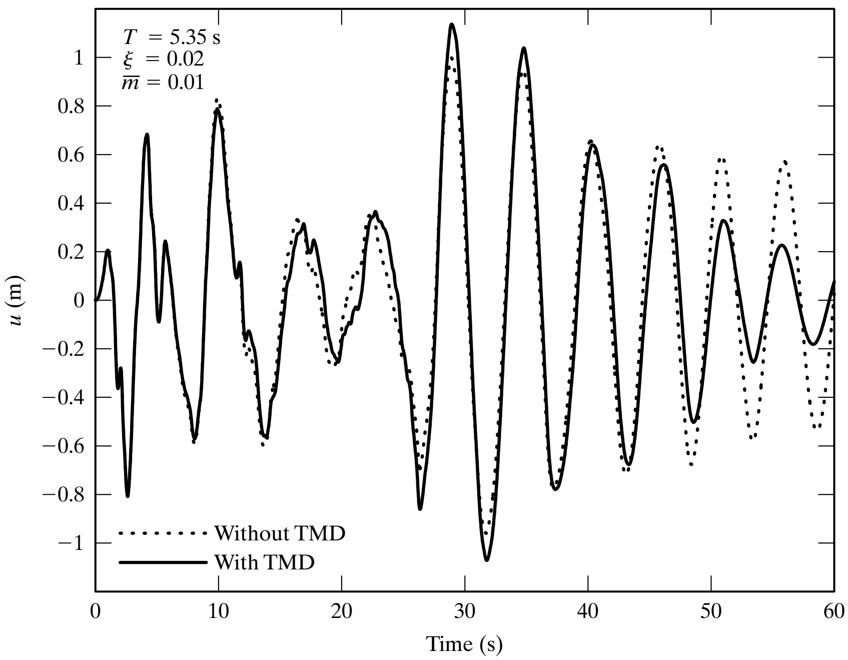


FIGURE 4.41: Response of SDOF to El Centro excitation.

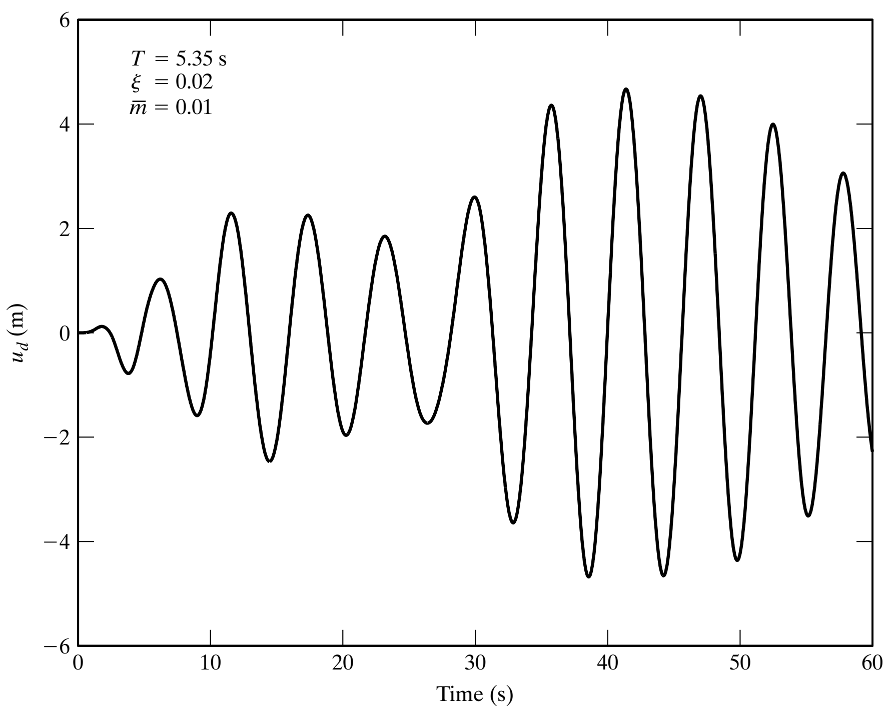


FIGURE 4.42: Relative displacement of TMD under El Centro excitation.

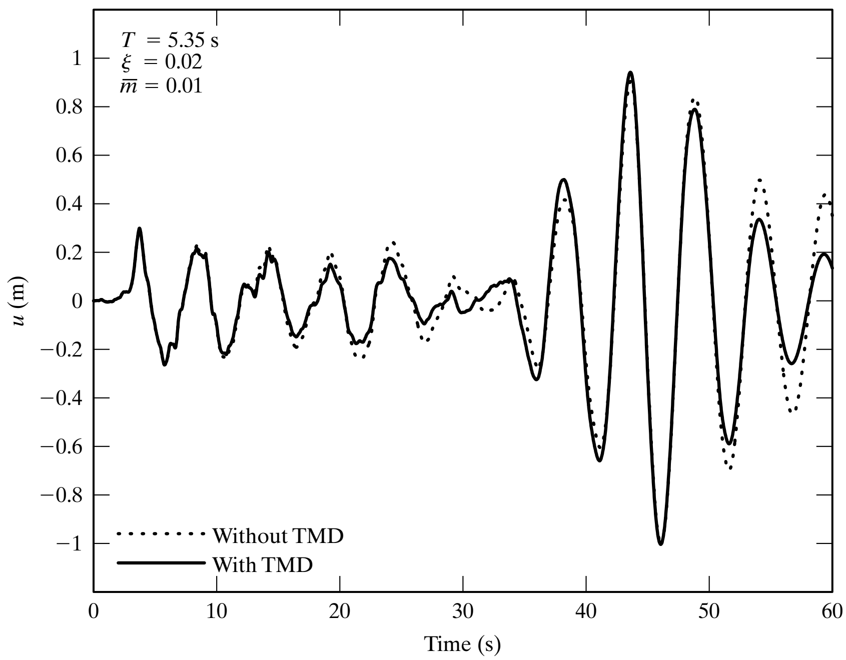


FIGURE 4.43: Response of SDOF to Taft excitation.

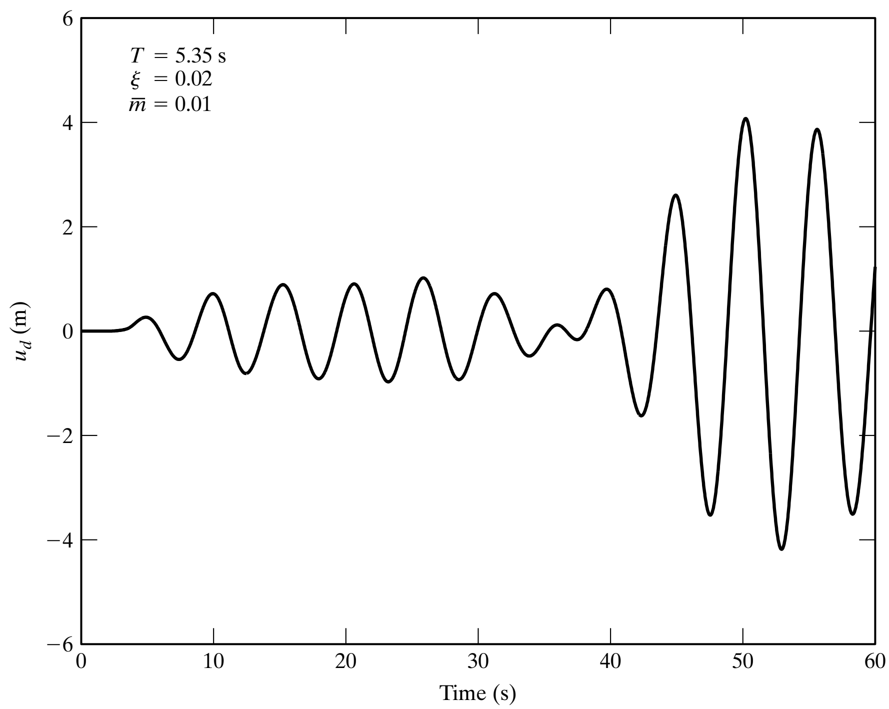


FIGURE 4.44: Relative displacement of TMD under Taft excitation.

4.6 TUNED MASS DAMPER THEORY FOR MDOF SYSTEMS

The theory of a SDOF system presented earlier is extended here to deal with a MDOF system having a number of tuned mass dampers located throughout the structure. Numerical simulations, which illustrate the application of this theory to the set of example building structures used as the basis for comparison of the different schemes throughout the text, are presented in the next section.

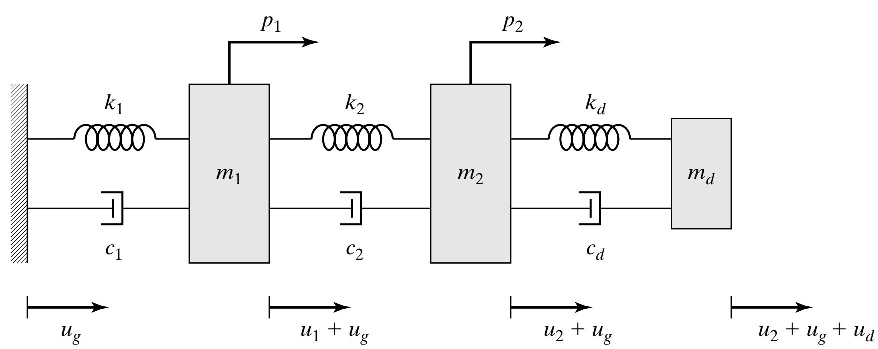


FIGURE 4.45: 2DOF system with TMD.

A 2DOF system having a damper attached to mass 2 is considered first to introduce the key ideas. The governing equations for the system shown in Figure 4.45 are

$$m_1\ddot{u}_1 + c_1\dot{u}_1 + k_1u_1 - k_2(u_2 - u_1) - c_2(\dot{u}_2 - \dot{u}_1) = p_1 - m_1\ddot{u}_g \tag{4.107}$$

$$m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) - k_du_d - c_d\dot{u}_d = p_2 - m_2\ddot{u}_g \tag{4.108}$$

$$m_d\ddot{u}_d + k_du_d + c_d\dot{u}_d = -m_d(\ddot{u}_2 + \ddot{u}_g) \tag{4.109}$$

The key step is to combine Eqs. (4.107) and (4.108) and express the resulting equation in a form similar to the SDOF case defined by Eq. (4.90). This operation reduces the problem to an equivalent SDOF system, for which the theory of Section 4.4 is applicable. The approach followed here is based on transforming the original matrix equation to scalar modal equations.

Introducing matrix notation, Eqs. (4.107) and (4.108) are written as

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \begin{bmatrix} p_1 - m_1a_g \\ p_2 - m_2a_g \end{bmatrix} + \begin{bmatrix} 0 \\ k_du_d + c_d\dot{u}_d \end{bmatrix} \tag{4.110}$$

where the various matrices are

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.111}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \tag{4.112}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{4.113}$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{4.114}$$

We substitute for $\mathbf{U}$ in terms of the modal vectors and coordinates

$$\mathbf{U} = \Phi_1 q_1 + \Phi_2 q_2 \tag{4.115}$$

The modal vectors satisfy the following orthogonality relations [see Eq. (2.211)]:

$$\Phi_j^T \mathbf{K} \Phi_i = \delta_{ij} \omega_j^2 \Phi_j^T \mathbf{M} \Phi_i \tag{4.116}$$

Defining modal mass, stiffness, and damping terms,

$$\tilde{m}_j = \Phi_j^T \mathbf{M} \Phi_j \tag{4.117}$$

$$\tilde{k}_j = \Phi_j^T \mathbf{K} \Phi_j = \omega_j^2 \tilde{m}_j \tag{4.118}$$

$$\tilde{c}_j = \Phi_j^T \mathbf{C} \Phi_j \tag{4.119}$$

expressing the elements of $\mathbf{\Phi}_j$ as

$$\Phi_j = \begin{bmatrix} \Phi_{j1} \\ \Phi_{j2} \end{bmatrix} \tag{4.120}$$

and assuming damping is proportional to stiffness

$$\mathbf{C} = \alpha \mathbf{K} \tag{4.121}$$

we obtain a set of uncoupled equations for q_1 and q_2:

$$\tilde{m}_j\ddot{q}_j + \tilde{c}_j\dot{q}_j + \tilde{k}_j q_j = \Phi_{j1}(p_1 - m_1 a_g) + \Phi_{j2}(p_2 - m_2 a_g + k_d u_d + c_d \dot{u}_d) \qquad j = 1, 2 \tag{4.122}$$

With this assumption, the modal damping ratio is given by

$$\xi_j = \frac{\tilde{c}_j}{2\omega_j \tilde{m}_j} = \frac{\alpha\omega_j}{2} \tag{4.123}$$

Equation (4.122) represents two equations. Each equation defines a particular SDOF system having mass, stiffness, and damping equal to $\tilde{m}$, $\tilde{k}$, and ξ. Since a TMD is effective for a narrow frequency range, we have to decide on which **modal** resonant response is to be controlled with the TMD. Once this decision is made, the analysis can proceed using the **selected** modal equation and the initial equation for the TMD [i.e., Eq. (4.109)].

Suppose the **first** modal response is to be controlled. Taking $j = 1$ in Eq. (4.122) leads to

$$\tilde{m}_1\ddot{q}_1 + \tilde{c}_1\dot{q}_1 + \tilde{k}_1 q_1 = \Phi_{11}p_1 + \Phi_{12}p_2 - [m_1\Phi_{11} + m_2\Phi_{12}]a_g + \Phi_{12}[k_d u_d + c_d\dot{u}_d] \tag{4.124}$$

In general, u_2 is obtained by superposing the modal contributions

$$u_2 = \Phi_{12}q_1 + \Phi_{22}q_2 \tag{4.125}$$

However, when the external forcing frequency is close to ω_1, the first mode response will dominate, and it is reasonable to assume

$$u_2 \approx \Phi_{12}q_1 \tag{4.126}$$

Solving for q_1

$$q_1 = \left[\frac{1}{\Phi_{12}}\right]u_2 \tag{4.127}$$

and then substituting in Eq. (4.124), we obtain

$$\tilde{m}_{1e}\,\ddot{u}_2 + \tilde{c}_{1e}\,\dot{u}_2 + \tilde{k}_{1e}\,u_2 = k_d u_d + c_d \dot{u}_d + \tilde{p}_{1e} - \Gamma_{1e}\,\tilde{m}_{1e}\,a_g \tag{4.128}$$

where $\tilde{m}_{1e}$, $\tilde{c}_{1e}$, $\tilde{k}_{1e}$, $\tilde{p}_{1e}$, and Γ_{1e} represent the **equivalent** SDOF parameters for the combination of mode 1 and node 2, the node at which the TMD is attached. Their definition equations are

$$\tilde{m}_{1e} = \left[\frac{1}{\Phi_{12}^2}\right]\tilde{m}_1 \tag{4.129}$$

$$\tilde{k}_{1e} = \left[\frac{1}{\Phi_{12}^2}\right]\tilde{k}_1 \tag{4.130}$$

$$\tilde{c}_{1e} = \alpha\tilde{k}_{1e} \tag{4.131}$$

$$\tilde{p}_{1e} = \frac{\Phi_{11}p_1 + \Phi_{12}p_2}{\Phi_{12}} \tag{4.132}$$

$$\Gamma_{1e} = \frac{\Phi_{12}}{\tilde{m}_1}(m_1\Phi_{11} + m_2\Phi_{22}) \tag{4.133}$$

Equations (4.109) and (4.128) are similar in form to the SDOF equations treated in the previous section. Both set of equations are compared next.

TMD equation

$$\begin{gathered} m_d\,\ddot{u}_d + c_d\,\dot{u}_d + k_d\,u_d = -m_d(\ddot{u} - a_g) \\ versus \\ m_d\,\ddot{u}_d + c_d\,\dot{u}_d + k_d\,u_d = -m_d(\ddot{u}_2 - a_g) \end{gathered} \tag{4.134}$$

Primary mass equation

$$\begin{gathered} m\ddot{u} + c\dot{u} + ku = c_d\,\dot{u}_d + k_d\,u_d + p - ma_g \\ versus \\ \tilde{m}_{1e}\,\ddot{u}_2 + \tilde{c}_{1e}\,\dot{u}_2 + \tilde{k}_{1e}\,u_2 = c_d\,\dot{u}_d + k_d\,u_d + \tilde{p}_{1e} - \Gamma_{1e}\tilde{m}_{1e}a_g \end{gathered} \tag{4.135}$$

Taking

$$\begin{gathered} u_2 \equiv u \qquad \tilde{m}_{1e} \equiv m \qquad \tilde{c}_{1e} \equiv c \qquad \tilde{k}_{1e} \equiv k \\ \tilde{p}_{1e} \equiv p \qquad \Gamma_{1e} \equiv \Gamma \end{gathered} \tag{4.136}$$

transforms the primary mass equation for the MDOF case to

$$m\ddot{u} + c\dot{u} + ku = c_d\, \dot{u}_d + k_d\, u_d + p - \Gamma m a_g \tag{4.137}$$

which differs from the corresponding SDOF equation by the factor Γ. Therefore, the solution for ground excitation generated earlier has to be modified to account for the presence of Γ.

The "generalized" solution is written in the same form as the SDOF case. We need only to modify the terms associated with a_g (i.e., H_6, H_8 and δ_6, δ_8). Their expanded form is as follows:

$$H_6 = \frac{\sqrt{[(\Gamma + \bar{m})f^2 - \Gamma\rho^2]^2 + [2\xi_d\rho f(\Gamma + \bar{m})]^2}}{|D_3|} \tag{4.138}$$

$$H_8 = \frac{\sqrt{[1 + \rho^2(\Gamma - 1)]^2 + [2\xi\rho]^2}}{|D_3|} \tag{4.139}$$

$$\tan a_2 = \frac{2\xi_d f\rho(\Gamma + \bar{m})}{f^2(\Gamma + \bar{m}) - \Gamma\rho^2} \tag{4.140}$$

$$\tan a_3 = \frac{2\xi\rho}{1 + (\Gamma - 1)\rho^2} \tag{4.141}$$

$$\delta_6 = a_2 - \delta_7 \tag{4.142}$$

$$\delta_8 = a_3 - \delta_7 \tag{4.143}$$

where $|D_3|$ is defined by Eq. (4.97), and δ_7 is given by Eq. (4.101).

From this point on, we proceed as described in Section 4.4. The mass ratio is defined in terms of the *equivalent* SDOF mass.

$$\bar{m} = \frac{m_d}{\tilde{m}_{1e}} \tag{4.144}$$

Given $\bar{m}$ and ξ_1, we find the tuning frequency and damper damping ratio using Figures 4.30 and 4.31. The damper parameters are determined with

$$m_d = \bar{m}\, \tilde{m}_{1e} \tag{4.145}$$

$$\omega_d = f_{\text{opt}}\omega_1 \tag{4.146}$$

$$c_d = 2\xi_d\big|_{\text{opt}}\,\omega_d m_d \tag{4.147}$$

Expanding the expression for the damper mass,

$$m_d = \bar{m}\,\tilde{m}_{1e} = \frac{\bar{m}[\Phi_1^T \mathbf{M} \Phi_1]}{\Phi_{12}^2} \tag{4.148}$$

shows that we should select the TMD location to coincide with the maximum amplitude of the mode shape that is being controlled. In this case, the first mode is the target mode, and Φ_{12} is the maximum amplitude for Φ_1.

This derivation can be readily generalized to allow for tuning on the ith modal frequency. We write Eq. (4.127) as

$$q_i \approx \left[\frac{1}{\Phi_{i2}}\right] u_2 \tag{4.149}$$

where i is either 1 or 2. The equivalent parameters are

$$\tilde{m}_{ie} = \left[\frac{1}{\Phi_{i2}^2}\right] \tilde{m}_i \tag{4.150}$$

$$\tilde{k}_{ie} = \omega_i^2 \tilde{m}_{ie} \tag{4.151}$$

Given $\tilde{m}_{ie}$ and ξ_i, we specify $\bar{m}$ and find the optimal tuning with

$$\omega_d = f_{\text{opt}} \omega_i \tag{4.152}$$

Example 4.4: Design of a TMD for a damped MDOF system

To illustrate the foregoing procedure, a 2DOF system having $m_1 = m_2 = 1$ is considered. Designing the system for a fundamental period of $T_1 = 1\text{ s}$ and a uniform deformation fundamental mode profile yields the following stiffnesses (refer to Example 1.6):

$$k_1 = 12\pi^2 = 118.44$$

$$k_2 = 8\pi^2 = 78.96$$

Requiring a fundamental mode damping ratio of 2%, and taking damping proportional to stiffness ($\mathbf{C} = \alpha\mathbf{K}$), the corresponding α is

$$\alpha = \frac{2\xi_1}{\omega_1} = \frac{0.02}{\pi} = 0.0064$$

The mass, stiffness, and damping matrices for these design conditions are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 197.39 & -78.96 \\ -78.96 & 78.96 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1.26 & -0.51 \\ -0.51 & 0.51 \end{bmatrix}$$

Performing an eigenvalue analysis yields the following frequencies and mode shapes:

$$\omega_1 = 6.28 \text{ rad/s} \qquad \omega_2 = 15.39 \text{ rad/s}$$

$$\Phi_1 = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix} \qquad \Phi_2 = \begin{bmatrix} 1.0 \\ -0.5 \end{bmatrix}$$

The corresponding modal mass, stiffness, and damping terms are

$$\tilde{m}_1 = \Phi_1^T \mathbf{M} \Phi_1 = 1.25 \qquad \tilde{m}_2 = \Phi_2^T \mathbf{M} \Phi_2 = 1.25$$

$$\tilde{k}_1 = \Phi_1^T \mathbf{K} \Phi_1 = 49.35 \qquad \tilde{k}_2 = \Phi_2^T \mathbf{K} \Phi_2 = 296.09$$

$$\tilde{c}_1 = \Phi_1^T \mathbf{C} \Phi_1 = 0.32 \qquad \tilde{c}_2 = \Phi_2^T \mathbf{C} \Phi_2 = 1.90$$

$$\xi_1 = \frac{\tilde{c}_1}{2\omega_1 \tilde{m}_1} = 0.02 \qquad \xi_2 = \frac{\tilde{c}_2}{2\omega_2 \tilde{m}_2} = 0.049$$

The optimal parameters for a TMD located at node 2, having a mass ratio of 0.01 and tuned to a specific mode, are as follows:

Mode 1: optimum location is node 2

$$f_{\text{opt}} = 0.982 \qquad \xi_d\big|_{\text{opt}} = 0.062$$

$$m_d = 0.0125 \qquad k_d = 0.4754 \qquad c_d = 0.0096$$

Mode 2: optimum location is node 1

$$f_{\text{opt}} = 0.972 \qquad \xi_d\big|_{\text{opt}} = 0.068$$

$$m_d = 0.0125 \qquad k_d = 2.7974 \qquad c_d = 0.0254$$

This result is for the damper located at node 2. When located at node 1, the mass and stiffness are reduced 75%.

The general case of a MDOF system with a tuned mass damper connected to the *n*th degree of freedom is treated in a similar manner. Using the notation defined previously, the *j*th modal equation can be expressed as

$$\tilde{m}_j\ddot{q}_j + \tilde{c}_j\dot{q}_j + \tilde{k}_j q_j = \tilde{p}_j + \Phi_{jn}[k_d u_d + c_d\dot{u}_d] \qquad j = 1, 2, \ldots \tag{4.153}$$

where $\tilde{p}_j$ denotes the modal force due to ground motion and external forcing, and Φ_{jn} is the element of Φ_j corresponding to the *n*th displacement variable. To control the *i*th modal response, we set $j = i$ in Eq. (4.153) and introduce the approximation

$$q_i \approx \left[\frac{1}{\Phi_{in}}\right]u_n \tag{4.154}$$

This leads to the following equation for u_n:

$$\tilde{m}_{ie}\,\ddot{u}_n + \tilde{c}_{ie}\,\dot{u}_n + \tilde{k}_{ie}\,u_n = \tilde{p}_{ie} + k_d u_d + c_d\dot{u}_d \tag{4.155}$$

where

$$\tilde{m}_{ie} = \left[\frac{1}{\Phi_{in}^2}\right]\tilde{M}_i = \left[\frac{1}{\Phi_{in}^2}\right]\Phi_i^T\mathbf{M}\Phi_i \tag{4.156}$$

$$\tilde{k}_{ie} = \omega_i^2\tilde{m}_{ie} \tag{4.157}$$

$$\tilde{c}_{ie} = \alpha\tilde{k}_{ie} \tag{4.158}$$

$$\tilde{p}_{ie} = \frac{1}{\Phi_{in}}\tilde{p}_i \tag{4.159}$$

The remaining steps are the same as described previously. We specify $\overline{m}$ and ξ_i, determine the optimal tuning and damping values with Figures 4.30 and 4.31, and then compute m_d and ω_d.

$$m_d = \overline{m}\;\tilde{m}_{ie} = \left[\frac{\overline{m}}{\Phi_{in}^2}\right]\Phi_i^T\mathbf{M}\Phi_i \tag{4.160}$$

$$\omega_d = f_{\text{opt}}\omega_i \tag{4.161}$$

The optimal mass damper for mode i is obtained by selecting *n* such that Φ_{in} is the maximum element in Φ_i.

Example 4.5: Design of TMDs for a simply supported beam

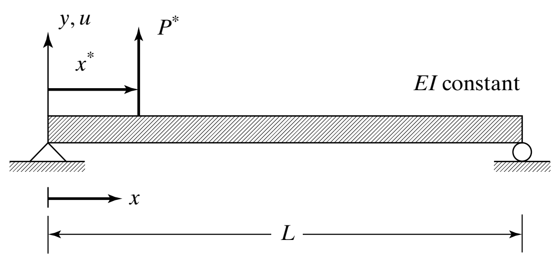


FIGURE E4.5a

Consider the simply supported beam shown in Figure E4.5a. The modal shapes and frequencies for the case where the cross sectional properties are constant and the transverse shear deformation is negligible are

$$\Phi_n(x) = \sin\frac{n\pi x}{L} \tag{1}$$

$$\omega_n^2 = \frac{EI}{\rho_m}\left(\frac{n\pi}{L}\right)^4 \tag{2}$$

$$n = 1, 2, \ldots$$

We obtain a set of N equations in terms of N modal coordinates by expressing the transverse displacement, $u(x, t)$, as

$$u = \sum_{j=1}^{N} q_i(t)\Phi_j(x) \tag{3}$$

and substituting for u in the principle of virtual displacements specialized for negligible transverse shear deformation [see Eq. (2.157)],

$$\int_0^L M\,\delta\chi\,dx = \int b\,\delta u\,dx \tag{4}$$

Substituting for $\delta\chi$,

$$\delta\chi = -\frac{d^2}{dx^2}(\delta u) \tag{5}$$

and taking

$$\delta u = \delta q_j \Phi_j \tag{6}$$

leads to the following equations:

$$-\int M\Phi_{j,\,xx}dx = \int b\Phi_j dx \qquad j = 1, 2, \ \ldots\ , N \tag{7}$$

Lastly, we substitute for M and b in terms of Φ and q and evaluate the integrals. The expressions for M and b are

$$M = EI\chi = -EI\sum_{l=1}^{N} q_l \Phi_{l,\,xx} \tag{8}$$

$$b = -\rho_m \ddot{u} + \bar{b}(x, t) = -\rho_m \sum_{l=1}^{N} \Phi_l \ddot{q}_l + \bar{b}(x, t) \tag{9}$$

Noting the orthogonality properties of the modal shape functions,

$$\int_0^L \Phi_j \Phi_k dx = \delta_{jk}\frac{L}{2} \tag{10}$$

$$\int_0^L \Phi_{j,\,xx}\Phi_{k,\,xx} dx = \left(\frac{j\pi}{L}\right)^4 \delta_{jk}\frac{L}{2} \tag{11}$$

the modal equations uncouple and reduce to

$$\tilde{m}_j \ddot{q}_j + \tilde{k}_j q_j = \tilde{p}_j \tag{12}$$

where

$$\tilde{m}_j = \frac{L\rho_m}{2} \tag{13}$$

$$\tilde{k}_j = EI\left(\frac{j\pi}{L}\right)^4 \frac{L}{2} \tag{14}$$

$$\tilde{p}_j = \int_0^L \bar{b} \sin\frac{j\pi x}{L} dx \tag{15}$$

When the external loading consists of a concentrated force applied at the location $x = x^*$ (see Figure E4.5a), the corresponding modal load for the jth mode is

$$\tilde{p}_j = P^* \sin \frac{j\pi x^*}{L} \tag{16}$$

In this example, the force is considered to be due to a mass attached to the beam as indicated in Figure E4.5b. The equations for the tuned mass and the force are

$$m_d(\ddot{u}^* + \ddot{u}_d) + k_d u_d + c_d \dot{u}_d = 0 \tag{17}$$

$$m_d(\ddot{u}^* + \ddot{u}_d) = -P^* \tag{18}$$

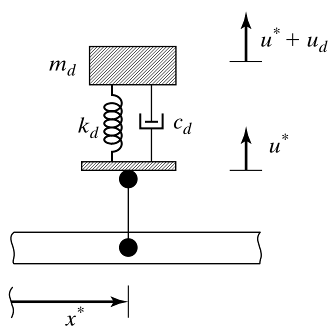


FIGURE E4.5b

Suppose we want to control the ith modal response with a tuned mass damper attached at $x = x^*$. Taking j equal to i in Eqs. (12) and (13), the ith modal equation has the form

$$\tilde{m}_i \ddot{q}_i + \tilde{k}_i q_i = (k_d u_d + c_d \dot{u}_d) \sin \frac{i\pi x^*}{L} \tag{19}$$

Assuming the response is dominated by the ith mode, $u^*(x^*, t)$ is approximated by

$$u^*(x^*, t) \approx q_i \sin \frac{i\pi x^*}{L} \tag{20}$$

and Eq. (19) is transformed to an equation relating u^* and u_d.

$$\tilde{m}_{ie} \ddot{u}^* + \tilde{k}_{ie} u^* = k_d u_d + c_d \dot{u}_d \tag{21}$$

where

$$\tilde{m}_{ie} = \frac{1}{\left(\sin\frac{i\pi x^*}{L}\right)^2}\tilde{m}_i \tag{22}$$

The remaining steps utilize the results generated for the SDOF undamped structure – damped TMD system considered in Section 4.3. We use $\tilde{m}_{ie}$ and $\tilde{k}_{ie}$ as the mass and stiffness parameters for the primary system.

To illustrate the procedure, consider the damper to be located at midspan, and the first mode is to be controlled. Taking $i = 1$ and $x^* = L/2$, the corresponding parameters are

$$\sin\frac{i\pi x^*}{L} = 1 \tag{23}$$

$$\tilde{m}_{ie} = \tilde{m}_1 = \frac{L\rho_m}{2} \tag{24}$$

$$\tilde{k}_{ie} = \tilde{k}_1 = \frac{EIL}{2}\left(\frac{\pi}{L}\right)^4 \tag{25}$$

We specify the equivalent damping ratio, ξ_e, and determine the required mass ratio from Figure 4.32. For example, taking $\xi_e = 0.06$ requires $\bar{m} = 0.03$. The other parameters corresponding to $\bar{m} = 0.03$ follow from Figures 4.29, 4.30, and 4.31.

$$f_{\text{opt}} = \frac{\omega_d}{\omega_1} = 0.965 \tag{26}$$

$$\xi_d\big|_{\text{opt}} = 0.105 \tag{27}$$

$$\frac{\hat{u}_d}{\hat{u}^*} = 5 \tag{28}$$

Using these parameters, the corresponding expression for the damper properties are

$$m_d = 0.03\tilde{m}_1 \tag{29}$$

$$\omega_d = 0.965\omega_1 \tag{30}$$

$$k_d = \omega_d^2 m_d \tag{31}$$

$$c_d = 2\xi_d\omega_d m_d \tag{32}$$

Once $\tilde{m}_1$ and ω_1 are specified, the damper properties can be evaluated. For example, consider the beam to be a steel beam having the following properties:

$$\begin{aligned} L &= 20 \text{ m} \\ \rho_m &= 1000 \text{ kg/m} \\ I &= 8 \times 10^{-4} \text{ m}^4 \\ E &= 2 \times 10^{11} \text{ N/m}^2 \end{aligned} \tag{33}$$

The beam parameters are

$$\begin{aligned} \tilde{m}_1 &= 10{,}000 \text{ kg} \\ \omega_1 &= 9.87 \text{ rad/s} \end{aligned} \tag{34}$$

Applying Eqs. (29) through (32) results in

$$\begin{aligned} m_d &= 300 \text{ kg} \\ \omega_d &= 9.52 \text{ r/s} \\ k_d &= 27{,}215 \text{ N/m} \\ c_d &= 599.8 \text{ N} \cdot \text{s/m} \end{aligned} \tag{35}$$

The total mass of the girder is 20,000 kg. Adding 300 kg, which is just 1.5% of the total mass, produces an effective damping ratio of 0.06 for the first mode response.

The mode shape for the second mode has a null point at $x = L/2$, and therefore locating a tuned mass at this point would have no effect on the second modal response. The optimal locations are $x^* = L/4$ and $x^* = 3L/4$. Taking $x^* = L/4$ and $i = 2$, we obtain

$$\sin\frac{i\pi x^*}{L} = 1 \tag{36}$$

$$\tilde{m}_{2e} = \tilde{m}_2 = \frac{L\rho_m}{2} \tag{37}$$

$$\tilde{k}_{2e} = \tilde{k}_2 = 8EIL\left(\frac{\pi}{L}\right)^4 \tag{38}$$

$$\omega_2^2 = \frac{16EI}{\rho_m}\left(\frac{\pi}{L}\right)^4 \tag{39}$$

The procedure from here on is the same as before. We specify ξ_e and determine the required mass ratio and then the frequency and damping parameters. It is of interest to compare the damper properties corresponding to the same equivalent damping ratio. Taking $\xi_e = 0.06$, the damper properties for the example steel beam are

$$m_d = 300\ \text{kg} \tag{40}$$

$$k_d = 435{,}440\ \text{N/m} \tag{41}$$

$$c_d = 2400\ \text{N}\cdot\text{s/m} \tag{42}$$

The required damper stiffness is an order of magnitude greater than the corresponding value for the first mode response.

4.7 CASE STUDIES: MDOF SYSTEMS

This section presents shear deformation profiles for the standard set of building examples defined in Table 2-4. A single TMD is placed at the top floor and tuned to either the first or second mode. The structures are subjected to harmonic ground acceleration with a frequency equal to the fundamental frequency of the buildings, as well as scaled versions of El Centro and Taft ground accelerations. As expected, significant reduction in the response is observed for the harmonic excitations (see Figures 4.46 through 4.49). The damper is generally less effective for seismic excitation versus harmonic excitation (see Figures 4.50 through 4.61). Results for the low period structures show more influence of the damper, which is to be expected since the response is primarily due to the first mode. This data indicates that a TMD is not the optimal solution for controlling the motion due to seismic excitation.

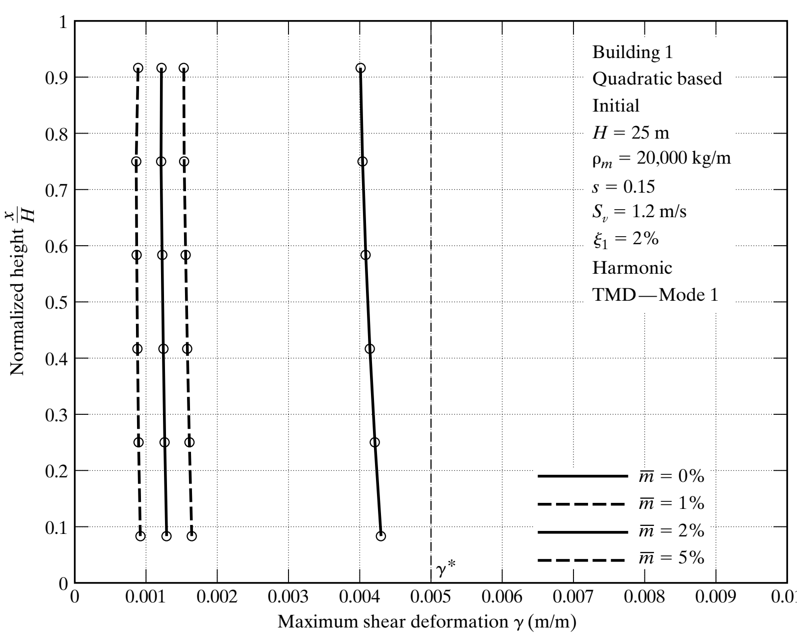


FIGURE 4.46: Maximum shear deformation for Building 1.

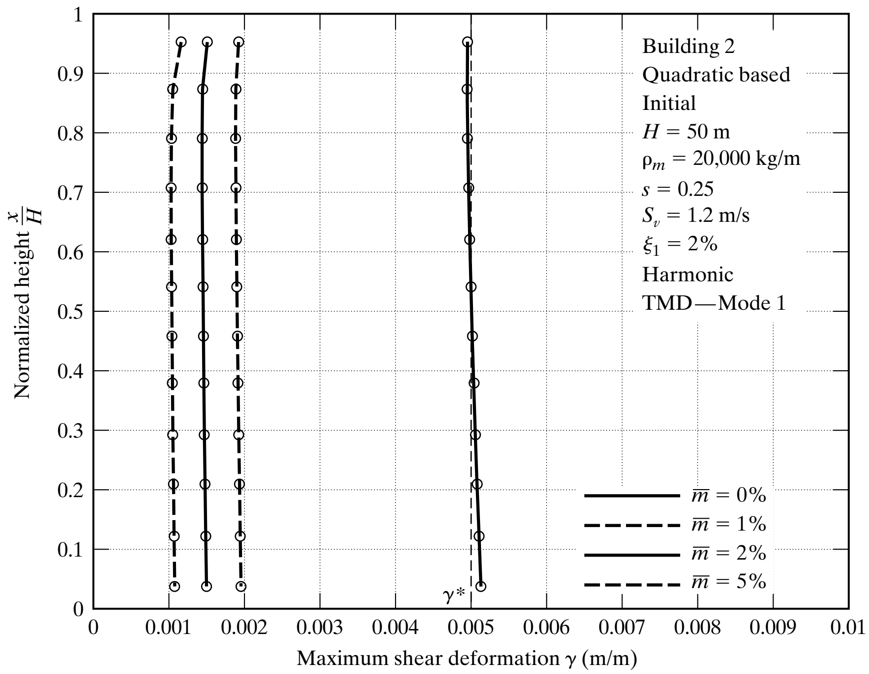


FIGURE 4.47: Maximum shear deformation for Building 2.

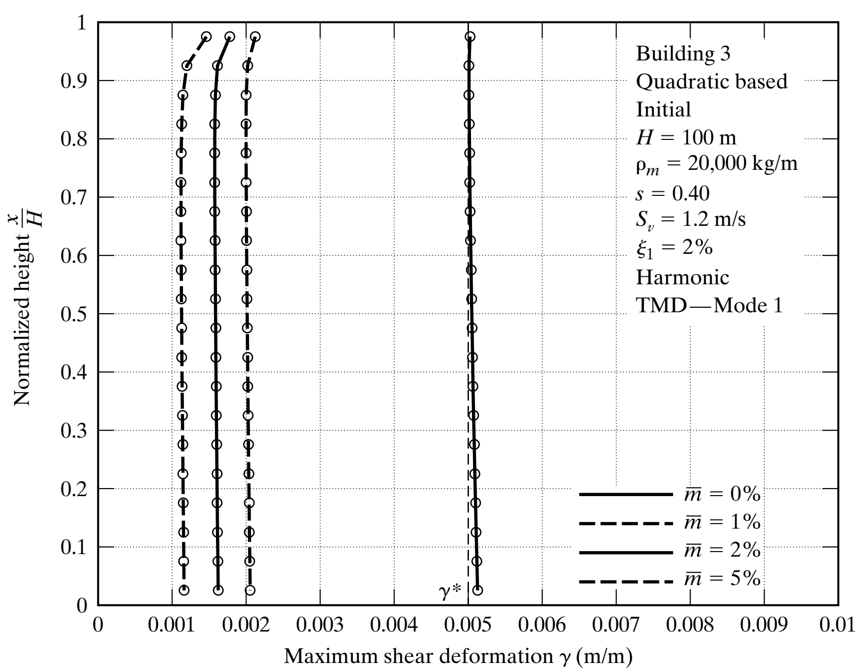


FIGURE 4.48: Maximum shear deformation for Building 3.

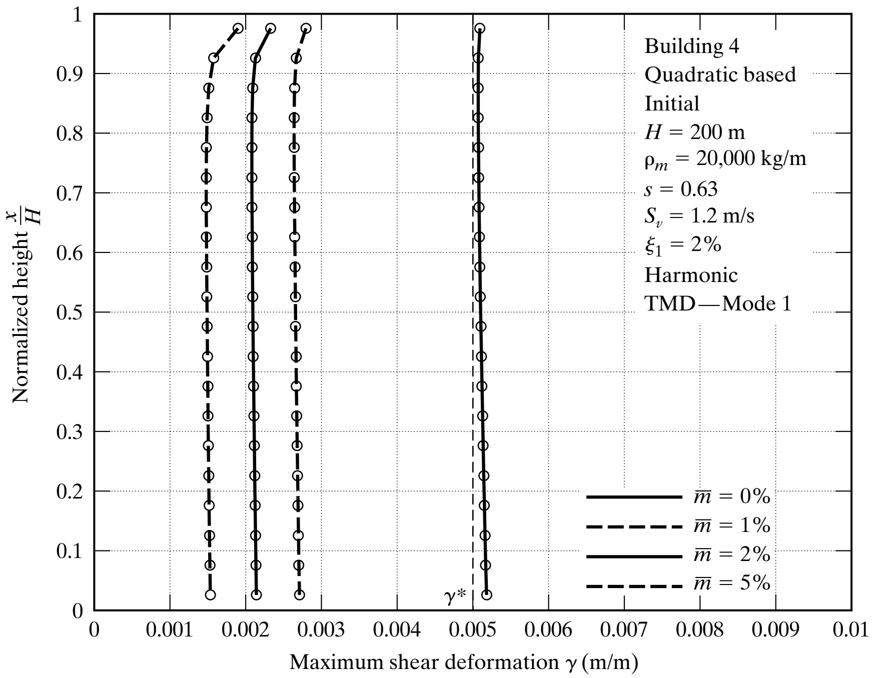


FIGURE 4.49: Maximum shear deformation for Building 4.

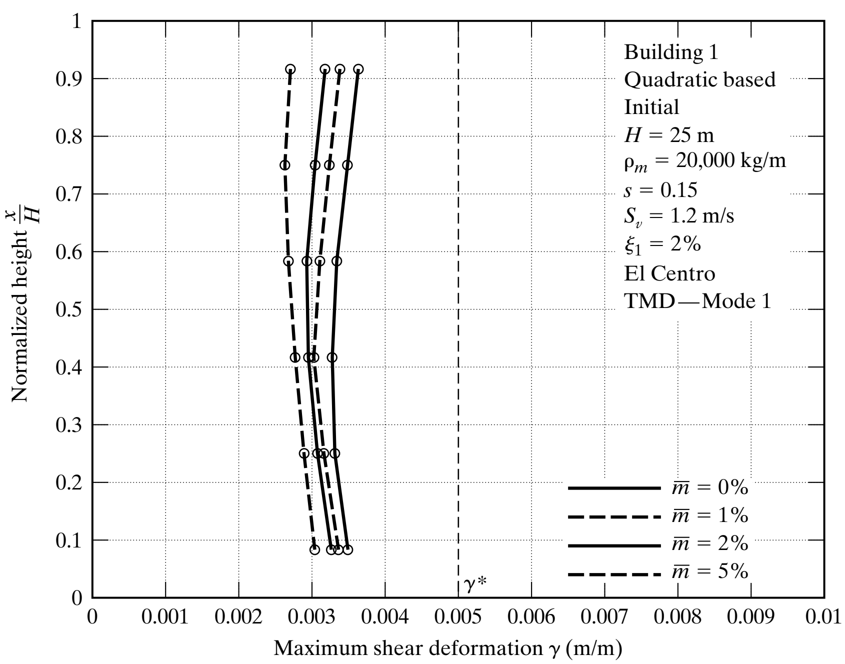


FIGURE 4.50: Maximum shear deformation for Building 1.

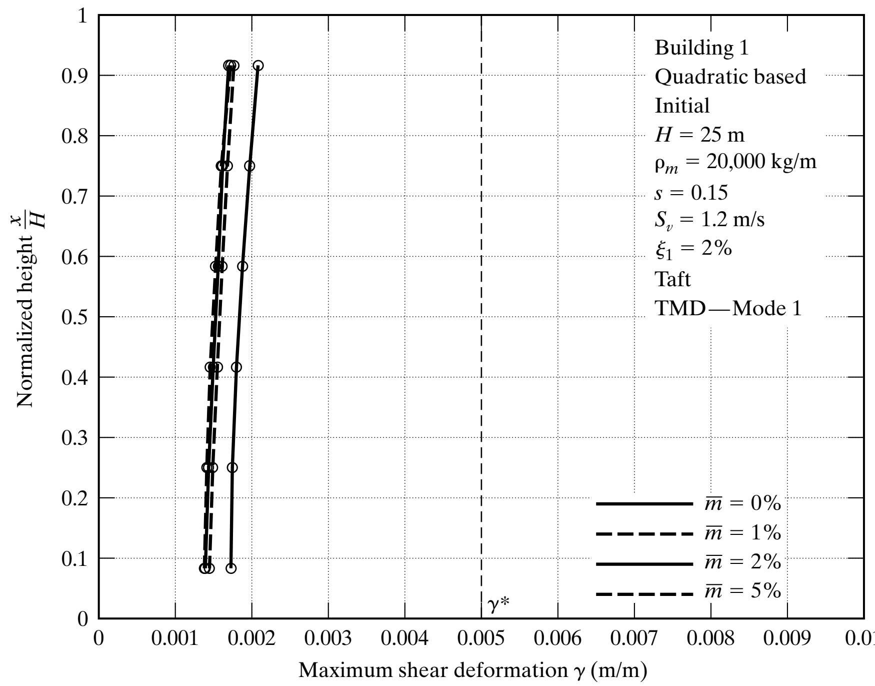


FIGURE 4.51: Maximum shear deformation for Building 1.

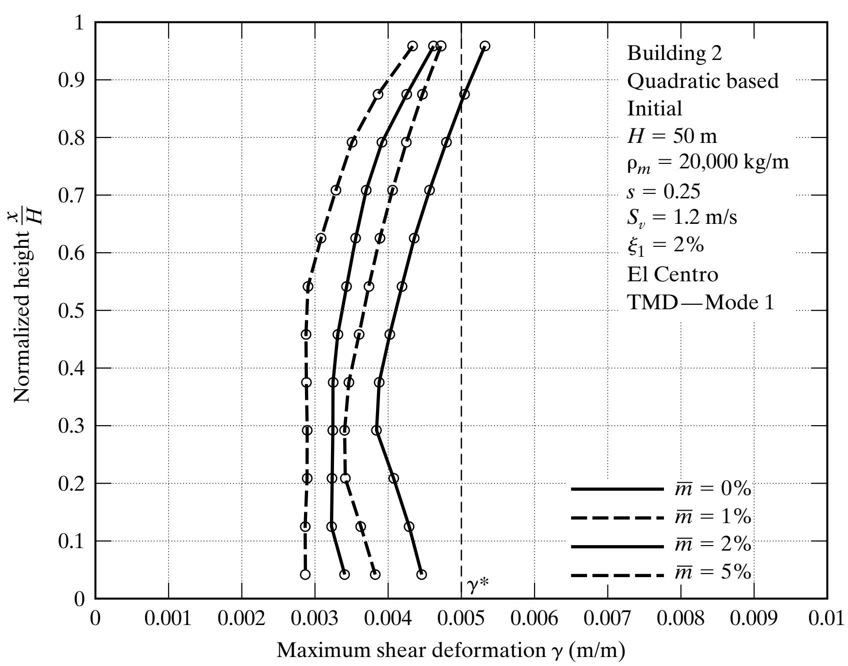


FIGURE 4.52: Maximum shear deformation for Building 2.

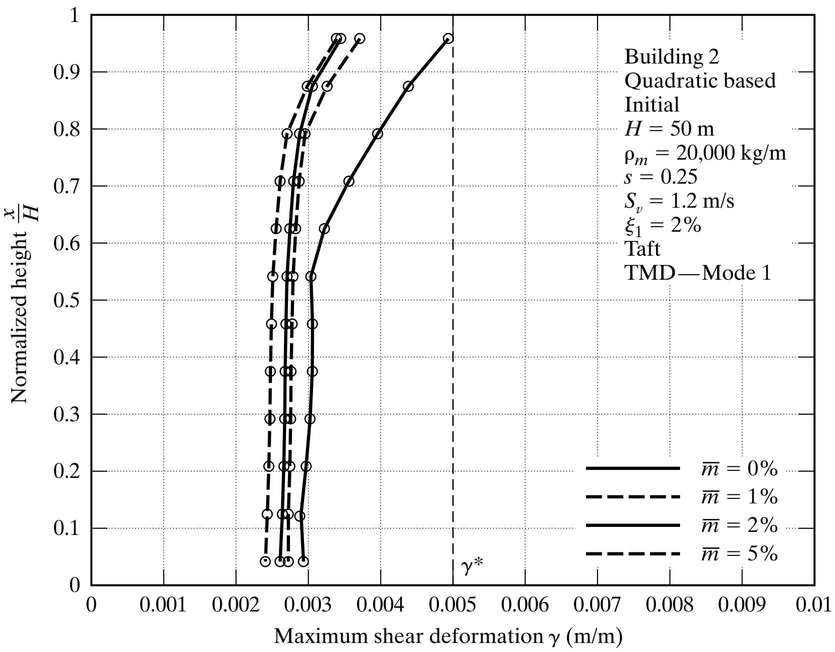


FIGURE 4.53: Maximum shear deformation for Building 2.

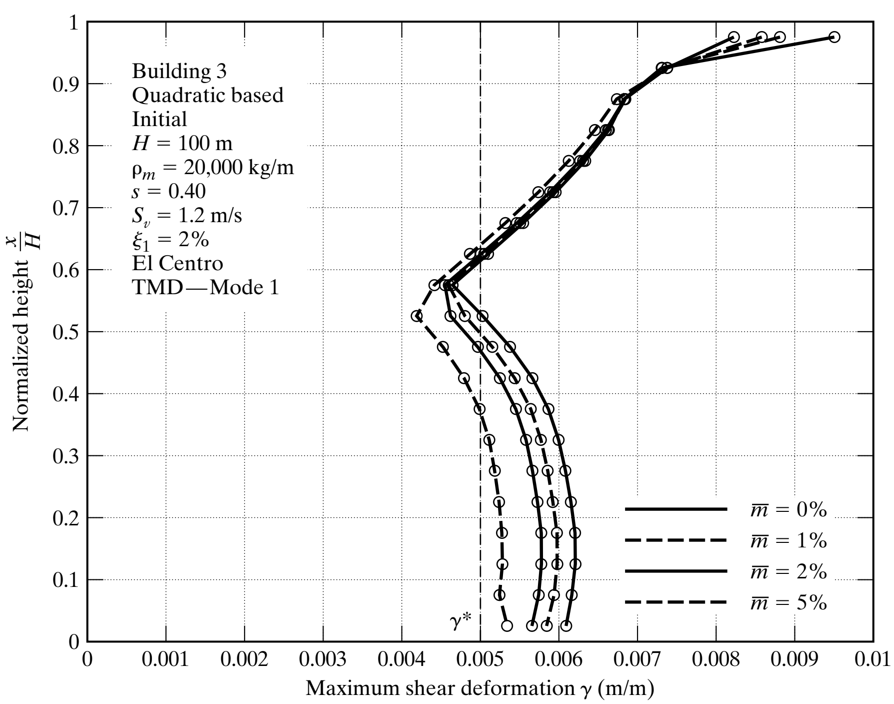


FIGURE 4.54: Maximum shear deformation for Building 3.

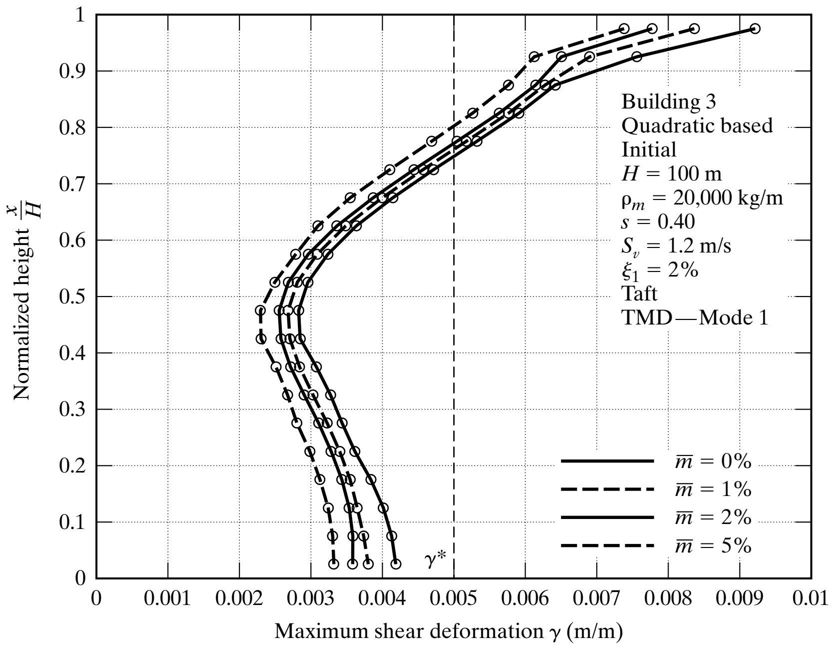


FIGURE 4.55: Maximum shear deformation for Building 3.

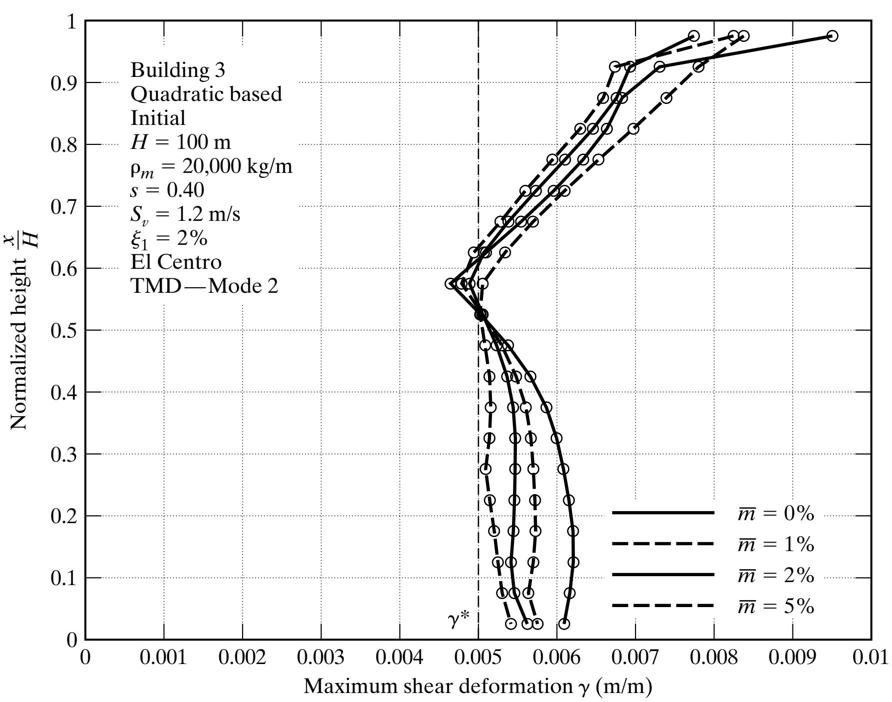


FIGURE 4.56: Maximum shear deformation for Building 3.

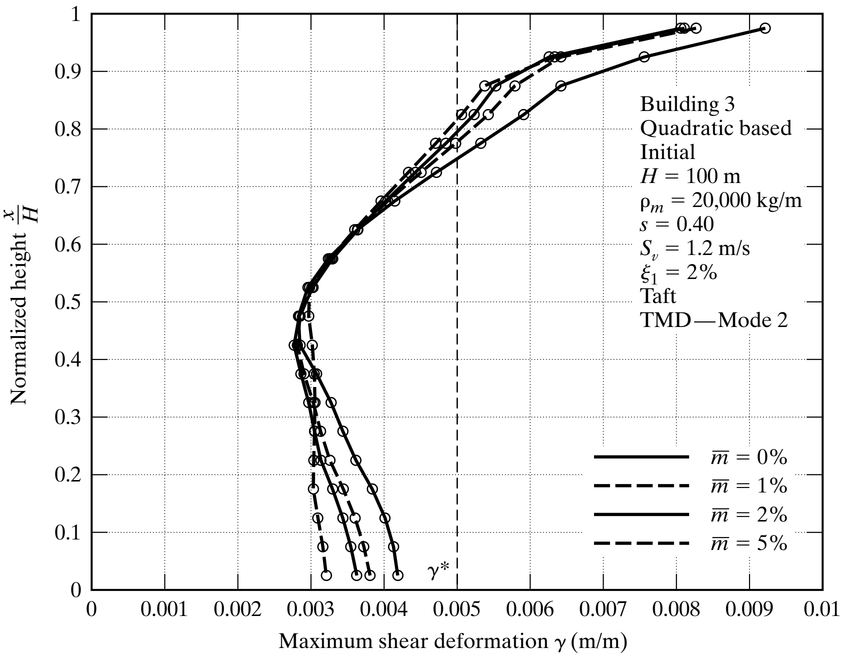


FIGURE 4.57: Maximum shear deformation for Building 3.

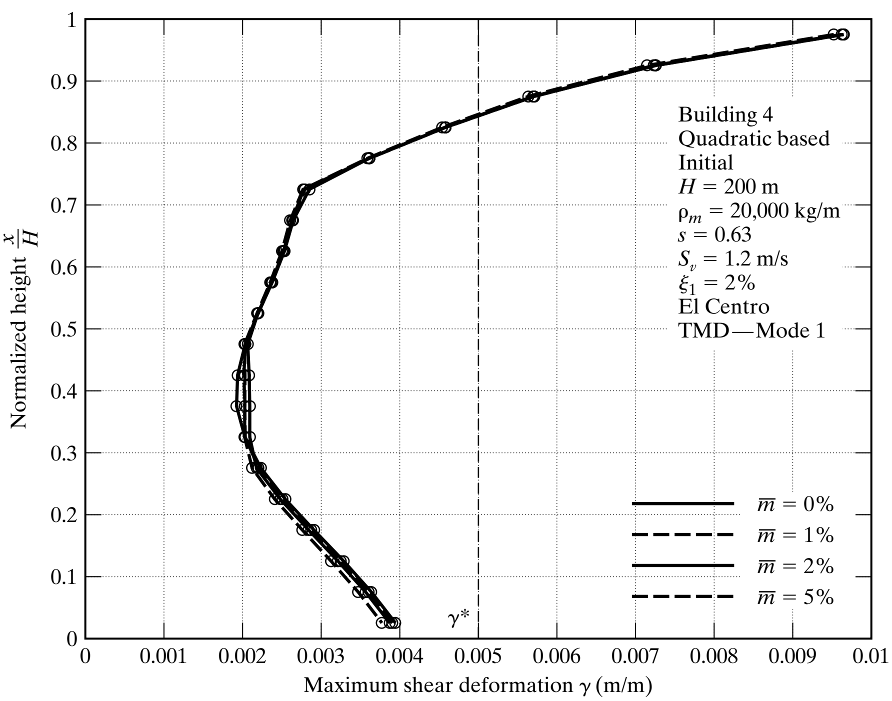


FIGURE 4.58: Maximum shear deformation for Building 4.

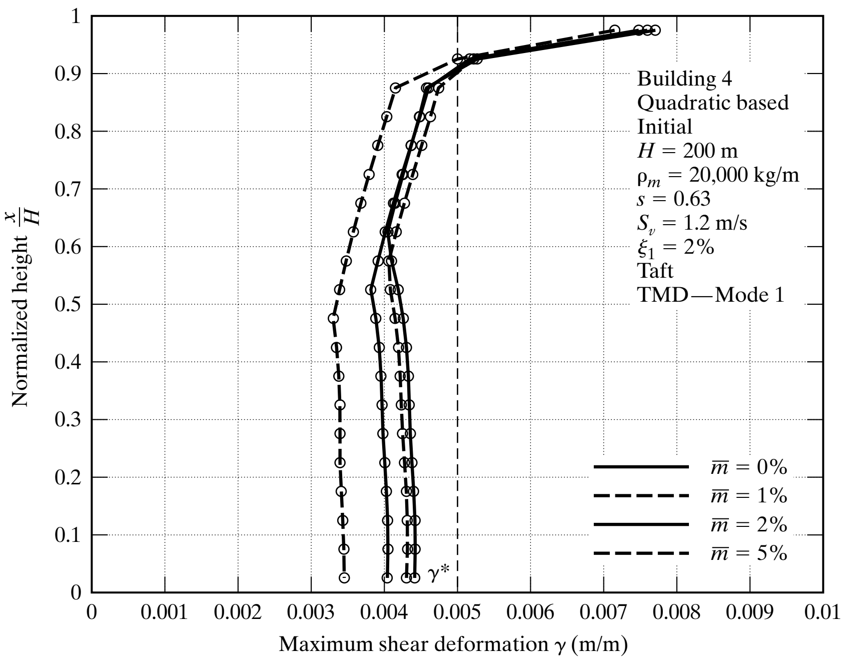


FIGURE 4.59: Maximum shear deformation for Building 4.

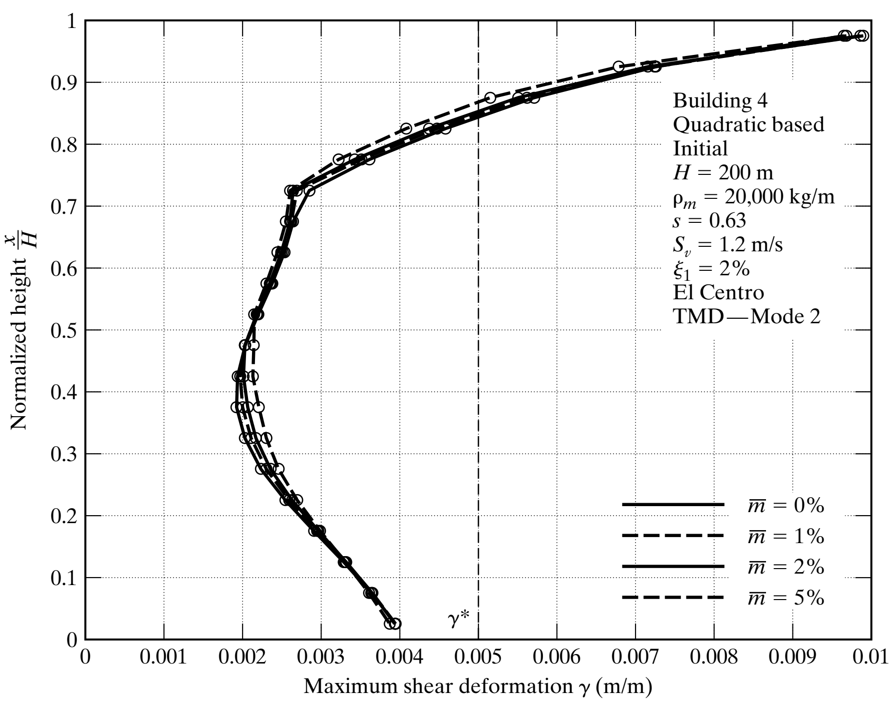


FIGURE 4.60: Maximum shear deformation for Building 4.

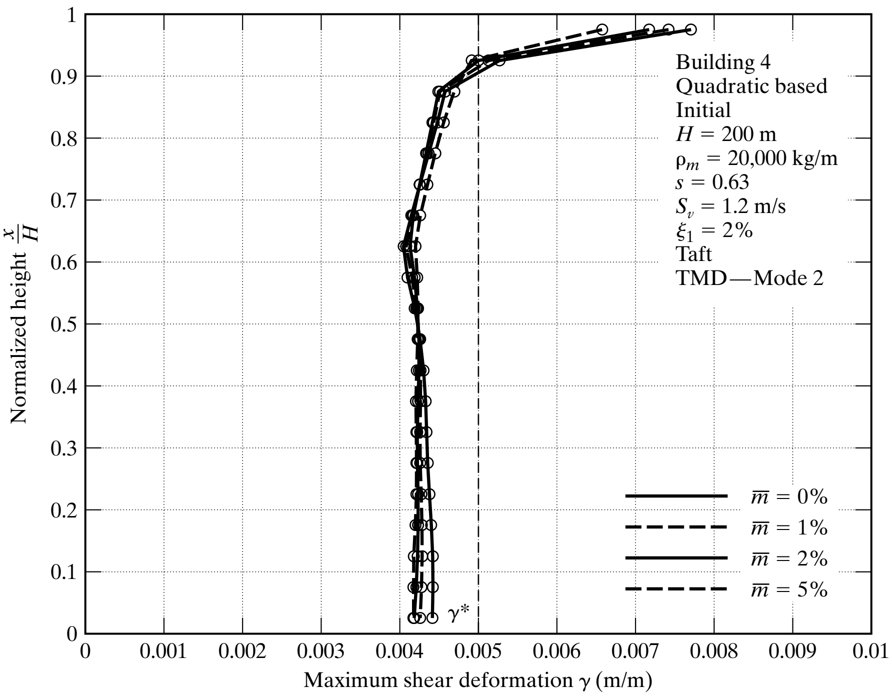


FIGURE 4.61: Maximum shear deformation for Building 4.

PROBLEMS

Problem 4.1

Verify Eqs. (4.13) through (4.17). *Hint*: Express p, u, and u_d in complex form

$$p = \hat{p}e^{i\Omega t}$$

$$u = \bar{u}e^{i\Omega t}$$

$$u_d = \bar{u}_d e^{i\Omega t}$$

and solve Eqs. (4.6) and (4.7) for $\bar{u}$ and $\bar{u}_d$. Then take

$$\bar{u} = \hat{u}e^{i\delta_1}$$

$$\bar{u}_d = \hat{u}_d e^{i(\delta_1 + \delta_2)}$$

$$\omega = \omega_d = \Omega$$

Problem 4.2

Refer to Eqs. (4.14) and (4.20). Express ξ_e as a function of $\overline{m}$, ξ, and $\hat{u}/\hat{u}_d$. Take $\xi = 0.05$, and plot ξ_e versus $\overline{m}$ for a representative range of the magnitude of the displacement ratio, $\hat{u}/\hat{u}_d$.

Problem 4.3

Figure 4.7 illustrates an active tuned mass damper configuration. The damper can be modeled with the 2DOF system shown in Figure P4.3. The various terms are as follows: u_s is the total displacement of the support attached to the floor beam; F_a is the self-equilibrating force provided by the actuator; m_d, k_d, c_d are parameters for the damper mass; k_a and m_a are parameters for the auxillary mass.

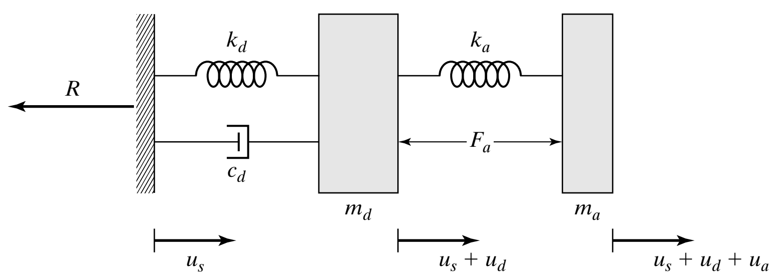


FIGURE P4.3

(a) Derive the governing equation for m_d and m_a. Also determine an expression for the resultant force, R, that the system applies to the floor beam.
(b) Consider m_a to be several orders of magnitude smaller than m_d (e.g., $m_a = 0.01 m_d$). Also take the actuator force to be a linear function of the relative velocity of the damper mass.

$$F_a = c_a \dot{u}_d$$

Specialize the equations for this case. How would you interpret the contribution of the actuator force to the governing equation for the damper mass?

Problem 4.4

Design a pendulum damper system having a natural period of 6 seconds and requiring less than 4 meters of vertical space.

Problem 4.5

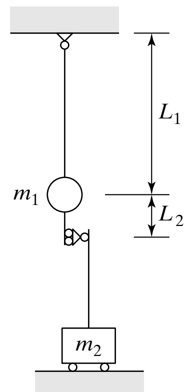


FIGURE P4.5

The pendulum shown in Figure P4.5 is connected to a second mass, which is free to move horizontally. The connection between mass 1 and mass 2 carries only shear. Derive an equation for the period of the compound pendulum and the length of an equivalent simple pendulum. Assume the links are rigid.

Problem 4.6

Refer to Figure 4.12. Establish the equations of motion for the mass, m_d, considering θ to be small. Verify that the equivalent stiffness is equal to W_d/R.

Problem 4.7

Refer to Figure 4.15 and Eq. (4.76). Derive the corresponding expression for $H_1|_{P,Q}$ starting with Eq. (4.62) and using the same reasoning strategy. Considering the mass ratio, $\bar{m}$, to be less than 0.03, estimate the difference in the "optimal" values for the various parameters.

Problem 4.8

Generate plots of H_1 versus ρ for ξ_d ranging from 0 to 0.2, $\bar{m} = 0.01$, and $f = 0.9876$. Compare the results with the plots shown in Figure 4.23.

Problem 4.9

Consider a system composed of an undamped primary mass and a tuned mass damper. The solution for periodic force excitation is given by [see Eqs. (4.52) to (4.71)]

$$u = \bar{u}e^{i\Omega t} \tag{1}$$

$$u_d = \bar{u}_d e^{i\Omega t} \tag{2}$$

$$\bar{u} = \frac{p}{k}H_1 e^{i\delta_1} \tag{3}$$

$$\bar{u}_d = \frac{p}{k}H_3 e^{i\delta_3} \tag{4}$$

$$H_1 = \frac{\sqrt{[f^2-\rho^2]^2+[2\xi_d\rho f]^2}}{|D_2|} \tag{5}$$

$$H_3 = \frac{\rho^2}{|D_2|} \tag{6}$$

$$|D_2| = \sqrt{([1-\rho^2][f^2-\rho^2]-\bar{m}\rho^2 f^2)^2+(2\xi_d\rho f[1-\rho^2(1+\bar{m})])^2} \tag{7}$$

The formulation for the optimal damper properties carried out in Section 4.3 was based on minimizing the peak value of H_1 (actually H_2 but H_1 behaves in a similar way) (i.e., on controlling the displacement of the primary mass). Suppose the design objective is to control the acceleration of the primary mass. Noting Eqs. (1) and (3), the acceleration is given by

$$\ddot{u} = a = \bar{a}e^{i\Omega t} \tag{8}$$

$$\bar{a} = \frac{p\Omega^2}{k}H_1 e^{i(\delta_1+\pi)} \tag{9}$$

Substituting for k transforms Eq. (9) to

$$\bar{a} = \frac{p}{m}H_1' \, e^{i(\delta_1+\pi)} \tag{10}$$

where

$$H_1' = \rho^2 H_1 \tag{11}$$

Investigate the behavior of H_1' with $\rho, f, \bar{m}$, and ξ_d. If it behaves similar to H_2, as shown in Figure 4.15, describe how you would establish the optimal values for the various parameters, and also how you would design a tuned mass system when H_1' is specified.

Problem 4.10

Design a TMD for a damped SDOF system having $\xi = 0.02$. The design motion constraints are

(a)

$$H_5\big|_{\text{opt}} < 10$$
$$\frac{H_7}{H_5\big|_{\text{opt}}} < 5$$

(b)

$$H_5\big|_{\text{opt}} < 5$$
$$\frac{H_7}{H_5\big|_{\text{opt}}} < 5$$

(c) Repeat part (b), considering ξ to be equal to 0.05.

Problem 4.11

This problem concerns the design of a tuned-mass damper for a damped single degree of freedom system. The performance criteria are

$$\xi_{\text{eq}} = 0.1 \qquad \hat{u}_d/\hat{u} = 5$$

(a) Determine the damper properties for a system having $m = 10{,}000$ kg and $k = 395$ kN/m for the following values of ξ:

- $\xi = 0.02$
- $\xi = 0.05$

(b) Will the damper be effective for an excitation with frequency 2.5π rad/s? Discuss the basis for you conclusion.

Problem 4.12

Refer to Example 3.7. Suppose a tuned mass damper is installed at the top level (at mass 5).

(a) Determine the damper properties such that the equivalent damping ratio for the fundamental mode is 0.16. Use the values of m, k, c from Example 3.7. Assume stiffness proportional damping for c.
(b) Consider the tuned mass damper to be a pendulum attached to m_5 (Figure P4.12). Determine m_d and L for the damper properties established in part (a).

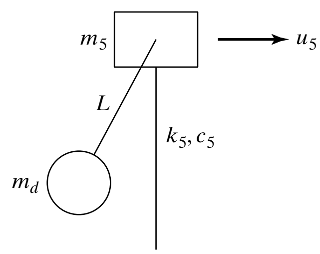


FIGURE P4.12

(c) Repeat part (a) for the case where the mass damper is tuned for the second mode rather than for the first mode, and the desired equivalent modal damping ratio is 0.3. Use the same values of m, k, c and assume stiffness proportional damping.

Problem 4.13

Consider a cantilever shear beam with the following properties:

- $H = 50$ m
- $\rho_m = 20{,}000$ kg/m
- $D_T = 8 \times 10^5\left(1 - \dfrac{0.6x}{H}\right)$ kN

(a) Model the beam as a 10DOF discrete shear beam having 5 m segments. Determine the first three mode shapes and frequencies. Normalize the mode shapes such that the peak amplitude is unity for each mode.
(b) Design tuned mass dampers to provide an effective modal damping ratio of 0.10 for the first and third modes. Take $\xi_1 = 0.02$ and assume modal damping is proportional to stiffness.

Note: You need to first establish the "optimal location" of the tuned mass dampers for the different modes.

Problem 4.14

Consider a simply supported steel beam having the following properties:

$$L = 30 \text{ m}$$

$$\rho_m = 1500 \text{ kg/m}$$

$$I = 1 \times 10^{-2} \text{ m}^4$$

(a) Design tuned mass damper systems that provide an equivalent damping of 0.05 for each of the first three modes.
(b) Repeat part (a) with the constraint that an individual damper mass cannot exceed 300 kg. *Hint*: Utilize symmetry of a particular mode shape to locate a pair of dampers whose function is to control that mode.

Problem 4.15

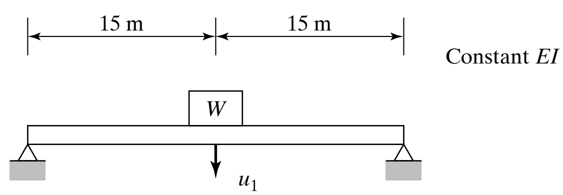


FIGURE 4.15

Consider the simply supported beam shown in Figure P4.15. The beam has a uniform weight of 15 kN/m and a concentrated weight at midspan of 100 kN. The flexural rigidity is constant and equal to 200,000 kN-m^2.

(a) Assume the first mode can be approximated by:

$$u = u_1 \sin\left(\frac{\pi}{L}x\right)$$

Determine the governing equation for u_1 using the principle of virtual displacements.
(b) Design a tuned mass damper to provide an equivalent damping ratio of 0.05 for the first mode. Assume no damping for the beam itself.
(c) Will the damper designed in part (b) be effective for the second mode? Explain your answer.

Problem 4.16

Refer to Problem 3.25, part (b). Suggest a tuned mass damper for generating the required energy dissipation.