AAE 607

VARIATIONAL PRINCIPLES

OF MECHANICS

LECTURES

by

Professor J. M. Longuski
AAE 607 VARIATIONAL PRINCIPLES OF MECHANICS

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AEE 607

Variational Principles of Mechanics

LECTURES

by

J.M. Longuski
VARIATIONAL PRINCIPLES OF MECHANICS

Instructor:

Prof. Longuski    Long-gús'-ski

Ph.D., U. of Mich., 1979 (Vinh and Greenwood)

1979-1982  Guidance & Control, JPL
1988-1992  Assistant Prof., Purdue
1992-1998  Associate Prof., Purdue
1998-present Professor, Purdue
AAE 607 VARIATIONAL PRINCIPLES OF MECHANICS

PROFESSOR J.M. LONGUSKI


Prerequisite: AAE 507 or equivalent.

Objectives: Graduate students in science and engineering will benefit from a strong background in the more abstract and intellectually satisfying areas of dynamical theory. Significant experience with Lagrange’s equation is assumed. The course will concentrate on the later discoveries of Hamilton and Jacobi which have resulted in variational principles of dynamics having unusual elegance and beauty. Also, Einstein’s General Theory of Relativity will be introduced.

TOPICS

- **Basic Concepts of Analytical Mechanics**
  
  Generalized coordinates, configuration space, nonholonomic constraints, virtual work, kinetic energy, and Riemannian geometry.

- **Calculus of Variations**
  
  Extremum problems, Euler-Lagrange equations, variation with auxiliary conditions, and nonholonomic conditions.

- **Hamilton’s Equations**
  
  Hamilton’s principle, Hamilton’s equations, modified Hamilton’s principle, least action, and phase space.

- **Hamilton-Jacobi Theory**
  
  Hamilton’s principal function, Pfaffian differential forms, Hamilton-Jacobi equation, Jacobi’s theorem, and separability.

- **Canonical Transformations**
  
  Generating functions, Lagrangian point transformations, canonical transformations, Lagrange and Poisson brackets, general transformations, and Liouville’s theorem.

- **Introduction to General Relativity**
  
Grading Policy

a) 25% on Term Paper

b) 75% on Your Journal (Notebook) which includes:
   i) Derivation of all reading material in Lanczos.
   ii) Derivation of lecture material.
   iii) Suggested problems in lecture.
   iv) Any other investigations you may wish to pursue.

On Campus Option

On campus students may volunteer to give a slide presentation. In this case the grading is as follows:

a) 15% on Term Paper
b) 10% on Presentation
c) 75% on Journal
1893 Born in Hungary
1921 Dissertation on quaternion treatment of special relativity
1928 Invited by Einstein to work with him on problems in general relativity
1931-1946 Professor of Mathematics and Physics at Purdue University
1946 Boeing Airplane Company; Lectured at University of Washington
1954 Senior Professor, School of Theoretical Physics, Dublin Institute
1968 Retires as Professor Emeritus (Dublin Institute of Advanced Studies)
1974 Dies in Budapest, Hungary at age of 81
Cornelius Lanczos contributed important and fertile ideas in an astonishingly broad range of disciplines, from relativity and quantum theory to applied mathematics and numerical analysis. He performed this work in an equally broad range of venues, from traditional academia to government laboratories to industry. The breadth of his interests stands as a monument in this era of extreme specialization.

Lanczos was born in Székesfehérvár, Hungary, in 1893, the son of a Jewish lawyer. In 1911 he entered the Hungarian Royal University of Budapest, graduating in 1916 after studies with Fejér and Eötvös, among others. He then became an assistant in the Department of Experimental Physics of the Technical University of Budapest and a doctoral candidate at the University of Szeged under Ortvay, a student of Sommerfeld. Lanczos wrote his dissertation on a quaternion treatment of special relativity and electrodynamics. In 1921 he became an assistant to Himstedt who was Director of the Institute for Theoretical Physics of the University of Freiburg. Lanczos left Freiburg in 1924 to work with E. Madelung, Director of the Institute for Theoretical Physics at the University of Frankfurt am Main; here Lanczos’s Habilitation took place in 1926 and he was named associate professor of theoretical physics in 1932. In the period 1922-1925, Lanczos published by himself over a dozen papers on relativity theory. While still at Frankfurt, in early 1926 he published a paper proposing a field-type (continuum) formulation of quantum mechanics using integral equations, anticipating Schrödinger’s equation. In 1928 Lanczos was invited by Einstein to come to Berlin to work with him; Lanczos maintained a correspondence with Einstein until the latter’s death in 1955, and continued to work on problems of relativity throughout his career.

Early in 1931, Lanczos accepted a one-year position at Purdue University, Indiana, as a visiting professor of mathematics and physics. This position became a permanent one, and Lanczos remained at Purdue, except for leaves of absence, until 1946. His research interests in nonrelativistic and relativistic quantum mechanics and general relativity were supplemented by a growing interest in the nascent field of numerical analysis. His first publication in this field, in 1938, put forth what is now widely known as the Lanczos Tau-method; subsequent work on Fourier analysis anticipated the now-ubiquitous Fast Fourier Transform. During the Second World War he spent a year at the National Bureau of Standards on the Mathematical Tables Project.

In 1946 Lanczos left Purdue to take up a position at the Boeing Airplane Company in Seattle, where he also lectured at the University of Washington. From 1949 to 1953, he worked at the Institute for Numerical Analysis, a project of the National Bureau of Standards at the University of California at Los Angeles. This position was followed by a year at North American Aviation. In 1954 Lanczos accepted the position of Senior Professor in the School of Theoretical Physics at the Dublin Institute for Advanced Studies, which he held until he retired in 1968 as Professor Emeritus. During this period he held visiting positions in various governmental departments, industries, and universities, including the Army Mathematical Research Center, Ford Motor Company, the University of North Carolina, the University of Michigan, and Yale. After his retirement, he remained based in Dublin, but traveled extensively to take up visiting appointments and lectureships at many U.S. and European universities. During the last two decades of his life, his research efforts in physics were primarily in gravitational theory and the underlying structure of space-time.

During a visit to Hungary in 1974, Lanczos died in Budapest at the age of 81.

Lanczos produced an enormous variety of fertile contributions to theoretical physics, applied mathematics, and numerical analysis. In theoretical and mathematical physics, he contributed pioneering ideas in relativity and quantum theory, as well as work in mechanics and electromagnetic theory. Lanczos’s contributions to mathematics include work on Chebyshev polynomials, Fourier series, Fourier analysis and synthesis, Fourier transforms, matrix eigenvalues and eigenfunctions, the gamma function, and ‘ill-posed’ problems. The American Mathematical Society awarded him the Chauvenet Prize in 1960 for a paper on the decomposition of an arbitrary rectangular matrix into three factors, the first and third being orthogonal and the other being diagonal. In the 1950’s and 1960’s, Lanczos’s numerical analysis work was very important for the development of efficient finite algorithms for numerical computation, as described in his book Applied Analysis (1956). Many of his contributions to applied mathematics and numerical analysis continue to play an important role in modern computational mathematics and physics. On the occasion of his 80th birthday, Lanczos was presented with a Festschrift, Studies in Numerical Analysis, with contributions from 19 eminent mathematicians and physicists.

Lanczos’s publications in mathematics and physics include over 100 papers and eight books. His books show clearly his viewpoint of the essential unity of mathematics and physics and their cultural appeal and importance. He had an outstanding ability to write physics and mathematics in a readable manner and his books, including the well-known Variational Principles of Mechanics, are milestones in the presentation of their subjects. Lanczos was generally recognized as being an outstanding teacher and expositor. His interests also extended beyond technical fields to encompass music and philosophy, and his bibliography includes several papers on art, theology, and the philosophy of science.

Lanczos made his first extended visit to North Carolina State University for a lecture series in 1965 as a guest of the School of Physical and Mathematical Sciences and the School of Design. He returned as a Visiting Professor of Mathematics and Physics in 1967 and 1968. His book entitled Space Through the Ages: The Evolution of Geometrical Ideas from Pythagoras and Euclid to Hilbert and Einstein (1970) was based on a lecture course given during the spring semester of 1968. As NCSU, Lanczos enjoyed a reputation not only as a distinguished mathematical physicist, but also as an accomplished pianist. Many faculty members from different areas of the fine arts, humanities, and sciences, present at the time, remember that Lanczos enjoyed discussing music, art, literature, and philosophy as much as he enjoyed talking about mathematics and physics.

The Cornelius Lanczos International Centenary Conference is one of several events highlighting this centennial year. On February 2, a Commemoration of Lanczos’s 100th birthday was held in Székesfehérvár, Hungary, featuring speeches on Lanczos’s work and the unveiling of a plaque marking the house where he was born. At North Carolina State University, the Conference will be accompanied by the publication of Lanczos’s Collected Papers, with English translations and commentaries. This multi-year effort has involved scholars all over the world as well as a major investment by North Carolina State University’s College of Physical and Mathematical Sciences.


PREFACE

The present treatise on the variational principles of mechanics should not be regarded as competing with the standard textbooks on advanced mechanics. Without questioning the excellent quality of these primarily technical and formalistic treatments, the author feels that there is room for monographs which exhibit the fundamental skeletons of the exact sciences in an elementary and philosophically oriented fashion.

7. Philosophical evaluation of the variational approach to mechanics. Although it is tacitly agreed nowadays that scientific treatises should avoid philosophical discussions, in the case of the variational principles of mechanics an exception to the rule may be tolerated, partly because these principles are rooted in a century which was philosophically oriented to a very high degree, and partly because the variational method has often been the focus of philosophical controversies and misinterpretations.

Chapter 1.
Basic Concepts of Analytical Mechanics

Newton's Laws Originally:
1. Applied to a single particle,
2. In an Inertial Ref. Frame.

Extensions: System of particles and kinematics (BKE)

Fortunately for Newton, bodies could be idealized as mass points (includes planets and other finite-sized bodies).
**Mechanical System**

Consider a system of $N$ particles, each of constant mass. For each particle:

$$\vec{F} = m\vec{a} = \vec{\dot{p}}$$

where

- $\vec{p} = m\vec{v}$ = linear momentum
- $\vec{F}$ = total force acting on the particle
- $\vec{v}$ = velocity of particle wrt some inertial frame

For the $i^{th}$ particle:

$$m_i\vec{\dot{v}}_i = \vec{F}_i + \vec{R}_i$$

where

- $\vec{F}_i$ = applied force (either a contact force or a field force)
- $\vec{R}_i$ = constraint force (workless in a virtual displacement and always a contact force).

**Contact Force** = Push or pull due to direct contact.

**Field Force** = Due to gravity or electrical fields.

---

**Vectorial Mechanics vs. Analytical Mechanics:**

1. VM isolates particles, AM considers whole.
2. VM constructs separate forces for each part.
   AM has one 'work f' (or PE).
3. VM must solve for constraint forces $\vec{R}_i$, AM uses constraint eqns.
4. AM gives entire set of EOM's from one unified principle (min. action), indep. of specific ref. sys.

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<th>VM</th>
<th>AM</th>
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<td>Isolates part's.</td>
<td>Considers whole.</td>
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<td>Sep. forces for ea.</td>
<td>One work f (or PE)</td>
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<td>Must solve for</td>
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<td>4.</td>
<td>Entire set of EOM's</td>
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<td>from uni. prin. (min. act.)</td>
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<td>indep. of spec. ref. sys.</td>
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**Generalized Coordinates**

A set of parameters whose numerical values specify the configuration of the system.

Generalization of the original idea of Descartes.

**DOF's (Degrees of Freedom)**

= number of coords. req'd to specify the config. minus the number of indep. eqns of constraint (EOC's).

**Indep. Gen. Coords.**

number of DOF's = number of gen.coords.

⇒ No constraints.
Transformation Eqns:

Consider the transformation from 3N Cartesian coords (rel. to inertial frame) to n gen. coords:

\[ x_i = x_i(q_1, q_2, ..., q_{3N}, t) \]
\[ x_2 = x_2(q_1, q_2, ..., q_{3N}, t) \]
\[ \vdots \]
\[ x_{3N} = x_{3N}(q_1, q_2, ..., q_{3N}, t) \]

Note that the transform eqns involve time explicitly.

EOC's

Possible for each sys. of coords \((x's \& q's)\) to have constraints:

If \(x's\) have \(l\) EOC's and \(q's\) have \(m\) EOC's

Then \(DOF's = 3N - l = n - m\)

To have one-to-one correspondence bet. \(x's\) and \(q's\) (for allowable domains),

or the nec. suf. cons. to solve \(q's\) as fn's of \(x's\) is

\[ \Delta \neq 0 \]  \hspace{1cm} (2)

where \(\Delta = \) Jacobian determinant of the transfr. (Eqs. 1)

Example

Suppose the 3N \(x's\) have \(l\) EOC's:

\[ f_j(x_1, x_2, ..., x_{3N}, t) = \alpha_j \]  \hspace{1cm} (3)
\[ j = (3N-l+1, ..., 3N) \]

Assume n inapp. gen. coorde are chosen so that

\[ n = 3N - l = DOF's \]  \hspace{1cm} (4)

Define an additional set of \(l\) \(q's\) and identify them with the \(l\) constant \(\alpha's\):

\[ q_j = \alpha_j \]  \hspace{1cm} (j = n+1, ..., 3N)  \hspace{1cm} (5)

Then the trans. eqns (1) can be written

\[ x_1 = x_1(q_1, q_2, ..., q_{3N}, t) \]
\[ x_2 = x_2(q_1, q_2, ..., q_{3N}, t) \]
\[ \vdots \]
\[ x_{3N} = x_{3N}(q_1, q_2, ..., q_{3N}, t) \]

(6)

If the Jacobian det. is nonzero:

\[ \Delta = \frac{\partial (x_1, x_2, ..., x_{3N})}{\partial (q_1, q_2, ..., q_{3N})} \neq 0 \]  \hspace{1cm} (7)

Then Eqs 1 or Eqs 6 can be solved for the \(q's\) as fn's of \(x's\) \& \(t\):

\[ q_j = f_j(x_1, x_2, ..., x_{3N}, t) \]  \hspace{1cm} (8)
\[ j = 1, 2, ..., n \]

The remaining constant \(q's\) \(q_{n+1}, ..., 3N\) were gurn by Eqs. 3 and 5.
Example of Transf. from Cart. to Gen. Coords.
Consider a particle constrained to move on a fixed circular path of radius $a$.

The EOC is
\[ (x_1^2 + x_2^2)^{1/2} = a \]  \hspace{1cm} (9)

Let a single gen. coord. $q_1$ represent the 1 DOF.
In accordance w Eq. 5, define a 2nd gen. coord. which is constant:
\[ q_2 = a \]  \hspace{1cm} (10)
The transf. eqns. are
\[ \begin{align*}
    x_1 &= q_2 \cos q_1 \\
    x_2 &= q_2 \sin q_1 
\end{align*} \]  \hspace{1cm} (11)
The Jacobian for this transf. is
\[ \frac{\partial (x_1, x_2)}{\partial (q_1, q_2)} = \begin{vmatrix}
    \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\
    \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2}
\end{vmatrix} = \begin{vmatrix}
    -q_2 \sin q_1 & \cos q_1 \\
    q_2 \cos q_1 & \sin q_1
\end{vmatrix} 
\]  \hspace{1cm} (12)

Hence, the $q$'s may be expressed as fins of the $x$'s except when $\Delta = 0 \rightarrow q_2 = 0$. (Radius = 0 $\rightarrow q_1$ undefined).
The transf. eqns are
\[ \begin{align*}
    q_1 &= \tan^{-1} \frac{x_2}{x_1} \\
    q_2 &= (x_1^2 + x_2^2)^{1/2}
\end{align*} \]  \hspace{1cm} (13)
where (we are taking):
\[ 0 \leq q_1 < 2\pi \\
0 \leq q_2 < \infty \]
in order that the $q$'s be single-valued fins of the $x$'s.
The TE's apply @ all pts on the finite $x_1 x_2$ plane except @ the origin.

Configuration Space
"The q space of n dimensions" (where we have n indep. gen. coords.) Explanation: If we are given n q's at a given time, we know the location of all the particles. We can think of the config. then, as a single point in n-space.

Example: Imagine 10 balls in 3-space that are indep. It takes 30 coords. to specify the config. This can be thought of as a single pt. in 30-space.
If the config. changes w/ time, then the config. pt., or C-pnt., traces a trajectory, or C-curve, in a q-space of n dimensions.

If there are any constraints of the form \( \phi_j(q_1, \ldots, q_n, t) = 0 \), then the C-pt. will be constrained to follow a hyperspace of less than n dimensions.

Example

3 dim's. \( (n = 3) \)

The point is constrained to motion on a 2-dimensional surface \( (2 < 3) \).

\[ x, y, z \]

Thus, we can write

\[ x = x'(x' + Vt') \quad (20) \]
\[ t = x'(t' + \frac{Vx'}{c^2}) \quad (21) \]

**Time dilation.** A clock fixed in I' at \( x'_i = x_i \) measures the time between two events: \( t'_i - t'_i' \). Using Eq. 20 we have

\[ t'_i - t'_i' = x' \left( t'_i - t'_i' + \frac{V}{c^2} (x'_i - x_i) \right) \quad (22) \]

\[ \Rightarrow \frac{t'_i - t'_i'}{t' - t} = \frac{1}{\gamma} (t'_i - t'_i') \]

\[ \Rightarrow \Delta t' < \Delta t \quad \text{for } V > 0 \quad (23) \]

The clock in I' is observed to run slower as seen in I (the lab frame).

**Example from Special Relativity**

Two inertial ref. frames in relative motion.

Then the L.T.'s (Lorentz Transformation Eq.) are:

\[ x^\prime = \gamma x (x - Vt) \quad (4) \]
\[ y^\prime = \gamma \frac{y}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (5) \]
\[ z^\prime = z (t - \frac{Vx}{c^2}) \quad (6) \]

Ignore the \( \phi \) and \( z \) coordinates, and compute the Jacobian determinant:

\[ \frac{\partial(x', y', z')}{\partial(x, y, z)} = \begin{vmatrix} \frac{x'}{x} & \frac{y'}{y} & \frac{z'}{z} \\ \frac{dx'}{dx} & \frac{dy'}{dy} & \frac{dz'}{dz} \\ \frac{dx}{x} & \frac{dy}{y} & \frac{dz}{z} \end{vmatrix} = \begin{vmatrix} -x & -y & x' \\ -y' & -x' & y' \end{vmatrix} \]

\[ = y^2 - x^2 = c^2 \left[ 1 - \frac{V^2}{c^2} \right] \quad (19) \]

\[ = 1 \neq 0 \Rightarrow x, y \text{ can always be expressed in terms of } x', y'. \]

**Longitudinal contraction.**

Let the rest length of the rod in I' be

\[ l = x_2 - x_1 \quad (25) \]

From Eq. 14 we have

\[ x_2 - x_1 = \frac{\gamma}{c^2} \left[ x_2 - x_1 - V(t_2 - t_1) \right] \quad (26) \]

\[ = 0 \text{ because we measure rod and "events" at same time in I}. \]

\[ l = \frac{\gamma}{c^2} l_0 \quad (27) \]

\[ \Rightarrow l = \frac{l}{l_0} \text{ for } V > 0 \]

The moving rod is measured to be shorter in the lab frame (I), than its rest length in I'.

\[ l < l_0 \quad \text{for } V > 0 \quad (29) \]
KINETIC ENERGY AND RIEMANNIAN GEOMETRY

Geometry of an n-dim space is determined by postulating the line element given by

$$ds^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} g_{ik} dx_i dx_k$$  \hspace{1cm} (1)

with the condition

$$g_{ik} = g_{ki}$$  \hspace{1cm} (2)

The special tensor $g_{ik}$ lies at the foundation of Riem. geo. and is called the **metrical tensor**. Cond. (2) => symmetric tensor.

---

If the Riemann curvature tensor ($\mathbf{R}$) is zero, then the space is Euclidean.

(Note: the components of $\mathbf{R}$ are $R_{\mu\nu\rho\sigma}$)

In special relativity

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

is the interval (between any two events)

$\Rightarrow$ Euclidean structure

In General Relativity,

$$ds^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} g_{ik} dq_i dq_k$$

we have 10 $g_{ik}$ which are $\frac{\partial^2 E}{\partial x_i \partial x_k}$, Gravitat. is interpreted as a geometrical phenomenon of the Riemannian structure of space time.

---

Advantages of Riem. Geo.

1. Geom. can be established analytically and indep. of any special ref. frame.
2. Eq. 1 is more general basis for building geo. (than Euclidean postulates).
   For Euc. geo., the $g_{ik}$ belong to certain class of $\mathbf{G}^{n_2}$. For general
   $g_{ik} \Rightarrow$ new geom., characterized by
   1. Properties of space change from pt. to pt. (continuously).
   2. Euc. geo. still holds in infinitesimal regions

---

Geodesics in Relativity*

The motion of a free particle can be expressed by the action principle

$$\int^{T_2}_{T_1} ds = 0$$

(Resembles Jacobi's principle w/o $V$.)

To find the motion of a force-free particle in this curvilinear ref. sys. we can solve the variational prob. with the Lagrangian

$$L = \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} g_{ik} q^i q^k}$$

where $\dot{q} = \frac{dq}{dt}$.

---

* See Lanczos, p. 329.
Kinetic Energy for Sys:

\[ T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{v}_i^2 \]  

(3)

where \( \dot{v}_i^2 = \left( \frac{d\mathbf{v}_i}{dt} \right)^2 = \frac{dx_i^2 + dy_i^2 + dz_i^2}{dt^2} \)  

(4)

Let us define the Line element for 3N-dim. space

\[ ds^2 = 2T dt^2 + \sum_{i=1}^{N} m_i \dot{v}_i^2 dt^2 \]

\[ = \sum_{i=1}^{N} \sum_{\alpha \in \{x, y, z\}} m_i \left( \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) \]  

(5)

Total KE:

\[ T = \frac{1}{2} m \left( \frac{ds}{dt} \right)^2 \]

(6)

with \( m = 1 \).

\( \Rightarrow \) KE of sys. may be replaced by KE of one particle of mass 1 in 3N dim. Config. space.

KE of sys. provides Riem. line elem. for Csp.

This is point in 3N Csp represents the entire mech. sys.

The mechanics of \( N \) free particles can be represented by the line element (5), which has Euclidean structure with coords. \( \sum_{i=1}^{N} x_i, \sum_{i=1}^{N} y_i, \sum_{i=1}^{N} z_i \).

Suppose we have \( m \) constraints.

\( \Rightarrow \) the particle is confined to a subspace of 3N-m dimensions, with Riemannian structure.

Jacobi's principle

Minimize the integral (action)

\[ A = \int_{t_0}^{t_1} \sqrt{2(E-V)} \, ds \]

(where \( ds \) is given by (5)) where

\( C \) is some parameter that must be chosen as the indend. variable (sometimes one of the \( q_i \)).

Mechanical prob. is translated into prob. of differential geometry.

CONSTRAINTS

2. Basic Types:

1. Holonomic.

Can be written in the form

\[ \phi_j \left( q_1, q_2, ..., q_m, \pi \right) = 0 \]

(7)

where \( j = 1, 2, ..., m \).

Example

2. particles connected by rigid rod of length \( l \):

EOC:

\[ (x-x_0)^2 + (y-y_0)^2 - l^2 = 0 \]

(8)

\( \Rightarrow \) 4 Cards - EOC = 3 DOF's
3 Nonholonomic

Nonintegrable differential expressions of the form:

\[ \sum_{i=1}^{n} a_i \dot{q}_i + \dot{q}_0 = 0 \quad i = 1, \ldots, m \]  \hspace{1cm} (9)

where \( a_i \)'s are \( f \)'s of \( q \)'s and \( t \).

See Lanczos for a beautiful example of a ball rolling without slipping.

Example

\[ \begin{array}{cc}
\text{If the disk can slip} & \Rightarrow 4 \text{ DOF's: } x, y, \phi, \rho (\text{all}) \\
\text{If disk cannot slip (along path or transverse to path)} & \Rightarrow 2 \text{ EOC} \Rightarrow 2 \text{ DOF's} \\
\text{Differential elem. along path:} & ds = r d\phi \\
\text{EOC's:} & dx - r \sin \alpha \, d\phi = 0 \\
& dy - r \cos \alpha \, d\phi = 0 \] \hspace{1cm} (10)

Why can't Eqs. 10 be integrated? Because if they could \Rightarrow

1. Eliminate \( \phi \) and \( \rho \) as \( f \)'s of \((x, y)\)
2. Orientation \((x, \phi)\) for given position \((x_t, y_t)\) is path independent (But we realize this can't be true.)
Consider the differential form
\[ P_1 dq_1 + P_2 dq_2 + P_3 dq_3 = 0 \]  
(1)
where \( P_1, P_2 \) and \( P_3 \) are functions of \( q_1, q_2, \) and \( q_3 \).

We wish to find an integrating factor \( \mu (q_1, q_2, q_3) \) such that
\[ \mu (P_1 dq_1 + P_2 dq_2 + P_3 dq_3) = \mu dq = 0 \]
(2)
The necessary and sufficient conditions that \( \mu \) and \( \phi \) exist (that \( (2) \) is integrable) are
\[ \frac{\partial}{\partial q_3} (\mu P_3) - \frac{\partial}{\partial q_2} (\mu P_2) = \frac{\partial}{\partial q_2} (\mu P_1) - \frac{\partial}{\partial q_1} (\mu P_2) \]
(3)
\[ \frac{\partial}{\partial q_1} (\mu P_3) - \frac{\partial}{\partial q_3} (\mu P_1) = \frac{\partial}{\partial q_3} (\mu P_2) - \frac{\partial}{\partial q_1} (\mu P_3) \]
(4)
\[ \frac{\partial}{\partial q_2} (\mu P_3) - \frac{\partial}{\partial q_1} (\mu P_2) = \frac{\partial}{\partial q_1} (\mu P_3) - \frac{\partial}{\partial q_2} (\mu P_3) \]
(5)

Thus, from Eq. (3):
\[ \mu \left( \frac{2P_3}{2q_3} - \frac{2P_2}{2q_2} \right) = P_2 \frac{2\phi}{2q_1} - P_1 \frac{2\phi}{2q_4} \]
(6)
and similarly, from Eqs. 4 and 5
\[ \mu \left( \frac{2P_3}{2q_3} - \frac{2P_1}{2q_1} \right) = P_2 \frac{2\phi}{2q_2} - P_1 \frac{2\phi}{2q_1} \]
(7)
\[ \mu \left( \frac{2P_2}{2q_2} - \frac{2P_3}{2q_3} \right) = P_3 \frac{2\phi}{2q_3} - P_1 \frac{2\phi}{2q_4} \]
(8)

Multiplying Eqs. (6), (7), and (8) by \( P_2, \)
P_1, and \( P_2, \) respectively, and adding we obtain

\[ \text{(For the RHS :)} \]
\[ P_2 \left( \frac{2P_3}{2q_3} - \frac{2P_2}{2q_2} \right) + P_1 \left( \frac{2P_3}{2q_3} - \frac{2P_1}{2q_1} \right) + P_3 \left( \frac{2P_3}{2q_3} - \frac{2P_2}{2q_2} \right) = 0 \]
\[ \Rightarrow P_1 \left( \frac{2P_3}{2q_3} - \frac{2P_2}{2q_2} \right) + P_2 \left( \frac{2P_3}{2q_3} - \frac{2P_1}{2q_1} \right) + P_3 \left( \frac{2P_3}{2q_3} - \frac{2P_2}{2q_2} \right) = 0 \]
(9)

**Side Note**

Note from the definition of curl \( \nabla \times F \) (where \( F = (P_1, P_2, P_3) \)) that
\[ \nabla \times F = \left( \frac{\partial}{\partial q_2} - \frac{\partial}{\partial q_3} \right) \vec{e}_1 + \left( \frac{\partial}{\partial q_3} - \frac{\partial}{\partial q_1} \right) \vec{e}_2 + \left( \frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \right) \vec{e}_3 \]
(10)
and thus
\[ \vec{F} \cdot \nabla \times \vec{F} = 0 \]  
(11)
is equivalent to Eq. 9, where
\[ \nabla \equiv \vec{e}_1 \frac{\partial}{\partial q_1} + \vec{e}_2 \frac{\partial}{\partial q_2} + \vec{e}_3 \frac{\partial}{\partial q_3} \]
(12)
\( \nabla \) is the nabla operator, introduced by Hamilton.

Note that we have interpreted \( q_1, q_2, q_3 \)
as Cartesian coordinates.

**End Side Note**
To have the same result as Lanczos' Example, let
\[
\begin{align*}
P_3 &= -1 \\
P_2 &= B_1 \\
Q &= B_2
\end{align*}
\] (10)

Substituting (10) into (9):
\[
B_1 \left( \frac{\partial B}{\partial q_3} - 0 \right) + B_2 \left( 0 - \frac{\partial B}{\partial q_3} \right) - \left( \frac{\partial B}{\partial q_2} - \frac{\partial B}{\partial q_3} \right) = 0
\] (11)

\[
\frac{\partial B}{\partial q_2} \frac{\partial B}{\partial q_3} = \frac{\partial B}{\partial q_2} + B_2 \frac{\partial B}{\partial q_3}
\] (12)

Same as Lanczos.

Example: Illustrate the Integibility Conditions

Use Lanczos' example:
\[
x dy + (y^2 - x^2 - z) dx + (z - y^2 - xy) dz = 0
\] (13)

In this case \( j = 1 \) and
\[
a_1 = y^2 - x^2 - z, \quad a_2 = 2 y^2 - xy, \quad a_3 = 2 x
\] (14).

Check exactness conditions:
\[
\frac{\partial a}{\partial y} = 2 y \neq \frac{\partial a}{\partial x} = -y
\] (15)

\[
\frac{\partial a}{\partial x} = -1 \neq \frac{\partial a}{\partial y} = 1
\] (16)

\[
\frac{\partial a}{\partial y} = 1 \neq \frac{\partial a}{\partial x} = 0
\] (17)

Thus Eq. 13 is not an exact differential.

Using Eq. 9 as the criterion, we have:
\[
P_1 = y^2 - x^2 - z, \quad P_2 = 2 y^2 - xy, \quad P_3 = x
\] (17)

\[
\frac{\partial P_1}{\partial y} = 2 y, \quad \frac{\partial P_2}{\partial y} = 2 y \neq \frac{\partial P_1}{\partial x} = -y
\] (18)

\[
\frac{\partial P_2}{\partial x} = 1, \quad \frac{\partial P_3}{\partial y} = -1 \neq \frac{\partial P_2}{\partial x} = 1
\] (19)

\[
\frac{\partial P_3}{\partial y} = 2 y, \quad \frac{\partial P_3}{\partial x} = -y \neq \frac{\partial P_3}{\partial x} = x
\] (20)

Summing the RHS of Eqs. 18 - 20 provides condition (9):
\[
\begin{align*}
& + y^2 - x^2 - y^2 + 2 y^2 - 2 y - 2 x y + 2 x y - 3 x y = 0 \quad (8) \\
& \Rightarrow -y^2 - x^2 + y + x y = 0
\end{align*}
\]

Or:
\[
\frac{2}{3} = x^2 - x y + y^2
\] (21)

Thus, Eq. 22 provides the holonomic constraint eq

We note that condition (9)

provides a more general test of interegability and provides the solution
in holonomic form.
Last time we derived the necessary and sufficient condition for the integrability of the ode.
\[ P_1 dq_1 + P_2 dq_2 + P_3 dq_3 = 0 \quad (1) \]
so that
\[ P_1 \left( \frac{dp_1}{dq_1} - \frac{dp_2}{dq_2} \right) + P_2 \left( \frac{dp_2}{dq_2} - \frac{dp_3}{dq_3} \right) + P_3 \left( \frac{dp_3}{dq_3} - \frac{dp_1}{dq_1} \right) = 0 \quad (2) \]

Notes:
Non-integrable ⇒ nonholonomic
⇒ No \( \phi : (q_1, q_2, q_3) \) exists such that \( d\phi = E_q (1) = 0 \)
⇒ Cannot use \( \phi (q) \) to eliminate some of the variables.

Notes: your HW is to prove Lanzos. Write your book for yourself.

Ice Skate Problem

Use Eq. (2):
\[ P_1 \left( \frac{dp_1}{dq_1} - \frac{dp_2}{dq_2} \right) + P_2 \left( \frac{dp_2}{dq_2} - \frac{dp_3}{dq_3} \right) + P_3 \left( \frac{dp_3}{dq_3} - \frac{dp_1}{dq_1} \right) = 0 \]

where
\[ \cos \theta \, dx + \sin \theta \, dy + \theta \, d\theta = 0 \]

and
\[ P_1 = \cos \theta \quad P_2 = \sin \theta \quad P_3 = 0 \]
\[ a_1 = x \quad a_2 = y \quad a_3 = \theta \]

\[ \Rightarrow \cos \theta \left[ \frac{2 \sin \theta}{y} - \frac{2 \theta}{y^2} \right] + \sin \theta \left[ \frac{2 \cos \theta}{x} - \frac{2 \theta}{x \cos \theta} \right] = 0 \]

\[ = 0 \]

\[ \Rightarrow \cos^2 \theta + \sin^2 \theta = 0 \]

\[ 1 = 0 \quad \text{not true} \]

⇒ nonholonomic

Another Classification of Constraints
Sleronomic Constraint \( EOC \not= \phi (t) \)
Rheonomic Constraint \( EOC = \phi (t) \)
Sleronomic System - characterized by \( x = x(q(t)) \)
(1) Transf. eqn. don't contain time explicitly
(2) Any EOC's are scleronomic.
Lanzos says if KE is scleronomic, momentum conservation is conserved.
Accessibility - If a system has
\[ 1 \ EOC \ \phi (q_1, q_2, ..., q_n, t) = 0 \quad (holonomic) \]

Then path in \( q \)-space is constrained to hypersurface (from EOC) in \( q \)-domain space. Not all of the \( q \)-space is accessible to \( C \) point.

But, for nonholonomic EOC: the \( C \) pt. must move on a differential surface element. Because of non-integrability ⇒ surf. elem. does not integrate to form a complete surface.
⇒ Entire \( q \)-space is accessible.
For Scleronomic System "Total Energy" is Constant:
\[ E = T + V \]  
\[ (3) \]
Provided
\[ V = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} q_i + U \]  
\[ (5) \]
where
\[ U = U(q_i, \dot{q}_i, \ddot{q}_i, \ldots, \dddot{q}_i) \]  
\[ (6) \]
Proof of Constancy of \( E \):
Assuming (to be shown later, as consequence of (8))
\[ F_i = \frac{\partial U}{\partial q_i} - \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_i} \]  
\[ (7) \]
Then the work done is
\[ W = \sum_{i=1}^{n} F_i \frac{dq_i}{dt} \]
\[ F_i \text{ gen force} \]
\[ = \sum_{i=1}^{n} \left( \frac{\partial U}{\partial q_i} - \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_i} \right) \frac{dq_i}{dt} \]
\[ = \sum_{i=1}^{n} \left[ \frac{\partial U}{\partial q_i} \frac{dq_i}{dt} - \sum_{i=1}^{n} \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{q}_i} \right) \frac{dq_i}{dt} \right] \]  
\[ (7) \]
From (7) - (9):  
\[ W = \sum_{i=1}^{n} \left[ \frac{\partial U}{\partial q_i} \frac{dq_i}{dt} - \sum_{i=1}^{n} \left( \frac{\partial U}{\partial q_i} \frac{dq_i}{dt} - \int \frac{\partial U}{\partial q_i} dq_i \right) \right] \]  
\[ (8) \]
\[ = \sum_{i=1}^{n} \left( \frac{\partial U}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial U}{\partial q_i} dq_i \right) \]  
\[ \text{Exact differential} \]
\[ = \delta U(q_i, q_i') \]
\[ W = (U - \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} \frac{dq_i}{dt}) \bigg|_{t}^{t_i} = -V \bigg|_{t}^{t_i} \]  
\[ (10) \]
\[ V = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} q_i - U \]  
\[ (11) \]
Since we know
\[ W = \Delta T = -\Delta V \]  
\[ (12) \]
\[ T + V = T + V \]  
\[ (13) \]
\[ E = \text{constant} \]
Derivation of LT's (Lorentz Transformation, Eqn. 35.2)

Two postulates of SR (Special Relativity):
1) Laws of physics same in all inertial frames.
2) Speed of light same in all inertial frames.

Assume 0, 0' coincide at t = t' = 0 when a flash of light is emitted at the common origin.

Wavefront location in I:
\[ x^2 + y^2 + z^2 = c^2 t^2 \]
Wavefront in I':
\[ x'^2 + y'^2 + z'^2 = c^2 t'^2 \]

The equation is valid for independent values of \( x \) and \( t \).

Equating terms in \( x^2 \):
\[ a^2 - c^2 b^2 = 1 \]  
Equating terms in \( t^2 \):
\[ b^2 - c^2 t^2 = -c^2 \]  
Equating terms in \( x t \):
\[ ab - c^2 e f = 0 \]

We have 3 equations and 4 unknowns \((a, b, e, f)\).

Fourth equation:
\[ x = V t \]
which gives the position of the origin \( 0' \) in \( I' \).

Using Eqn. k in Eqn. c, we have
\[ x' = 0 = a V t + bt \]  
\[ x = 0' \]

The LT's must be consistent with Eqn. a and b. Also, since a free particle moves with constant velocity in a straight line, our any inertial frame \( I' \) transformations must be linear.

Clearly:
\[ y' = y \]  
\[ z' = z \]

Let us assume transformation Eqn. of the form
\[ x' = a x + b t \]  
\[ t' = c x + d t \]
where \( a, b, c, d \) are constants to be determined (that may depend on the constant \( V \)).

Using Eqs. c - f in Eqs. a and b, we have
\[ x^2 - c^2 t^2 = x'^2 - c^2 t'^2 \]
\[ = a^2 x^2 + 2 a b x t + b^2 t^2 - c^2 (c x^2 + 2 c d x t + d^2 t^2) \]

From Eqn. l,
\[ b = -V a \]

Similarly, since the position of 0 is at \( x = 0 \) or \( x' = V t' \), Eqn. e gives
\[ x' = -V t' = 0 + bt \]  
and Eqn. f gives:
\[ t' = 0 + bt \]

Substituting Eqn. 0 into Eqn. n provides
\[ -V t e = bt \]
\[ \Rightarrow b = -V \]

From Eqn. m and p we see that
\[ q = f \]

Substituting Eqn. p into Eqn. i gives
\[ V^2 t^2 - c^2 t^2 = -c^2 \]
\[ \Rightarrow \Delta^2 = \frac{c^2}{c^2 - V^2} \]
From Eqns $a$ and $r$ we have

$$a = \frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}}$$  \hspace{1cm} (8)

(We choose the positive root because $a$ and $f$ must approach unity for Galilean transformations as $V \to 0$.)

Using Eqns $m$ and $s$ we obtain

$$b = \frac{-V}{\sqrt{1 - \left(\frac{V}{c}\right)^2}}$$  \hspace{1cm} (9)

and from Eqns $f$ and $s$ we get

$$e = \frac{ab}{c^2} = \frac{b^2}{c^2} = \frac{-V/c^2}{\sqrt{1 - \left(\frac{V}{c}\right)^2}}$$  \hspace{1cm} (10)

Defining $\gamma = \frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}}$  \hspace{1cm} (11)

we can write the LTs using Eqns $e, f, s, t, u$ and $v$:

\begin{verbatim}
\begin{align*}
x' &= \gamma (x - Vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma (t - \frac{Vx}{c^2})
\end{align*}
\end{verbatim}

Lorentz Transformation Eqns.